Partial fractions for $\int \frac{f(x)}{g(x)}\,dx$

$f(x), g(x)$: polynomials

Step 1 Make sure $\deg f < \deg g$

Step 2: Find prime factors of g

$$(x-r_i)^{m_i},\ \left[(x-a_j)^2+b_j^2\right]^{n_j}$$

Step 3: $(x-r)^m \longrightarrow \frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_m}{(x-r)^m}$

$$((x-a)^2+b^2)^n \longrightarrow \frac{B_1x+C_1}{(x-a)^2+b^2}+\frac{B_2x+C_2}{((x-a)^2+b^2)^2}+\cdots+\frac{B_nx+C_n}{((x-a)^2+b^2)^n}$$

Eg 1. $\displaystyle\int \frac{x^4}{(x-1)^3}\,dx$

$$x^4 = x(x^3-3x^2+3x-1)$$

$$+3x^3-9x^2+9x-3$$

$$+6x^2-8x+3$$

$$\frac{x^4}{(x-1)^3} = x+3+\frac{6x^2-8x+3}{(x-1)^3}$$

$$\frac{6x^2-8x+3}{(x-1)^3} = \frac{A_1}{(x-1)}+\frac{A_2}{(x-1)^2}+\frac{A_3}{(x-1)^3}$$

$$\Rightarrow 6x^2-8x+3 = A_1(x-1)^2+A_2(x-1)+A_3$$

$x \leftarrow 1 \Rightarrow A_3 = 1$

$\frac{d}{dx},\ x \leftarrow 1 \Rightarrow A_2 = 4$

$\frac{d^2}{dx^2},\ x \leftarrow 1 \Rightarrow A_1 = 6$

<u>Check</u>.

$6 \quad -12 \quad 6$

$4 \quad -4 \quad = 6 - 8 \; 3 \quad \text{OK}$

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$$\text{Ans} = \int (x+3) + \frac{6}{(x-1)} + 4(x-1)^{-2} + (x-1)^{-3}$$

$$= \frac{x^2}{2} + 3x + 6\ln|x-1| - 4(x-1)^{-1} - \frac{1}{2}(x-1)^{-2} + C$$

Eg: $\int \frac{x^4}{(1+x^2)^2(x-1)}\,dx$

Sol:

$$= \frac{A_1}{(x-1)} + \frac{B_1x+C_1}{(1+x^2)} + \frac{B_2x+C_2}{(1+x^2)^2}$$

$$x^4 = A_1(1+x^2)^2 + (B_1x+C_1)(x-1)(1+x^2) + (B_2x+C_2)(x-1)$$

$$x \leftarrow 1 \Rightarrow A_1 = \frac{1}{4}$$

$$\Rightarrow \frac{1}{4}(3x^4-2x^2-1) = (x-1)\begin{pmatrix}(B_1x+C_1)(1+x^2) \\ + B_2x+C_2\end{pmatrix}$$

$$3x^3+3x^2+x+1 = 4\left((B_1x+C_1)(1+x^2)+B_2x+C_2\right)$$

Eg: $\displaystyle\int \frac{x^2+1}{(x-1)(x-2)(x-3)}\,dx$

Sol

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-2)} + \frac{A_3}{(x-3)}$$

$$(x^2+1) = A_1(x-2)(x-3) + A_2(x-1)(x-3) + A_3(x-1)(x-2)$$

$x \leftarrow 1 \Rightarrow 2 = 2A_1$

$x \leftarrow 2 \Rightarrow 5 = -A_2$

$x \leftarrow 3 \Rightarrow 10 = 2A_3$

$\therefore$ Ans

$= \ln|x-1| - 5\ln|x-2| + 5\ln|x-3| + C$

$$\frac{x^4}{(x-1)^2(x^2+1)^2} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{B_1x+C_1}{(1+x^2)} + \frac{B_2x+C_2}{(1+x^2)^2}$$

$$x^4 = A_1(x-1)(1+x^2)^2 + A_2(1+x^2)^2 + (B_1x+C_1)(x-1)^2(1+x^2) + (B_2x+C_2)(x-1)^2$$

$$x \leftarrow 1 \Rightarrow A_2 = \cdots \quad \therefore \frac{1}{4}(3x^4-2x^2-1) = (x-1)\Big(A_1(1+x^2)^2 + (B_1x+C_1)(1+x^2)(x-1) + (B_2x+C_2)(x-1)\Big)$$

$$\Rightarrow \frac{1}{4}(3x^3+3x^2+x+1) = A_1(1+x^2)^2 + (B_1x+C_1)(1+x^2)(x-1) + (B_2x+C_2)(x-1)$$

$x \leftarrow 1 \Rightarrow A_1 = \frac{1}{2} \Rightarrow$ proceed as previous example

Half angle substitution

$$\int \frac{P(\cos x, \sin x)}{Q(\cos x, \sin x)}\,dx$$

Let $t = \tan\frac{x}{2} \Rightarrow dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$$dx = \frac{2\,dt}{1+t^2}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\tan\frac{x}{2}\cos^2\frac{x}{2} = \frac{2t}{1+t^2}$$

$$\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

Reduces to $\int \frac{f(t)}{g(t)}\,dt$, f, g: polynomials of t

$$3x^3+3x^2+x+1=商(x^2+1)+餘$$

$$=(3x+3)(x^2+1)+(-2x-2)$$

$$Ans=\int \frac{\frac{1}{4}}{(x-1)}+\frac{\frac{1}{4}(3x+3)}{(1+x^2)}+\frac{\frac{1}{4}(-2x-2)}{(1+x^2)^2}dx$$

$$\int \frac{1}{(x-1)}dx \to \ln|x-1|$$

$$\int \frac{1}{1+x^2}dx \to \tan^{-1}x \quad , \quad \int \frac{x}{1+x^2}dx \to \frac{1}{2}\ln(1+x^2)$$

$$\int \frac{x}{(1+x^2)^2}dx \to \frac{-1}{2}(1+x^2)^{-1} \quad , \quad \int \frac{1}{(1+x^2)^2}dx \to \begin{matrix} x=\tan t \\ dx=\sec^2 t\,dt \end{matrix}$$