Recall

$$\int C^m S^{2l+1}dx = -\int C^m(1-C^2)^l dC$$

$$\int C^{2k+1}S^n dx = \int (1-S^2)^k S^n dS$$

If both m, n are even

$$\int C^m S^n dx = \frac{C^{m+1}S^{n-1}}{m+n} + \frac{n-1}{m+n}\int C^m S^{n-2}dx$$

$(m+n \neq 0)$

$$= \frac{C^{m-1}S^{n+1}}{m+n} + \frac{m-1}{m+n}\int C^{m-2}S^n dx$$

$(m,n) \to \begin{matrix}(m-2,n)\\(m,n-2)\end{matrix} \to \cdots \begin{matrix}(2,0)\\(0,2)\end{matrix}$ (*)

(*): $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Eg $\int \sin^2 x \cos^4 x \, dx$

Method 1: <u>double angle formula</u>

$$S^2 \to C_2, \quad C^4 \to C_2^2 \to C_4$$

involves integrals of $\sin 2x$, $\cos 2x$, $\cos^2 2x$, $\cos^3 2x$ (see textbook)

<u>Method 2</u>:

$$\int C^4 S^2 \, dx = \int C^3 S^2 \, dS = \frac{1}{3} \int C^3 \, dS^3$$

$$= \frac{1}{3}\left(C^3 S^3 - \int S^3 \, dC^3 \right) = \frac{1}{3}\left(C^3 S^3 + 3 \int S^3 C^2 S \, dx \right)$$

$$= \frac{1}{3}\left(C^3S^3 + 3\int S^4C^2\,dx\right)$$

$S^2 = 1 - C^2$

$$= \frac{1}{3}\left(C^3S^3 + 3\int S^2C^2\,dx - 3\int S^2C^4\,dx\right)$$

$$\Rightarrow 3\int C^4S^2\,dx = C^3S^3 + 3\int C^2S^2\,dx - 3\int S^2C^4\,dx$$

$$\therefore \int C^4S^2\,dx = \frac{C^3S^3}{6} + \frac{3}{6}\int C^2S^2\,dx \qquad \begin{matrix} m=4 \\ n=2 \end{matrix}$$

$$= \frac{C^3S^3}{6} + \frac{1}{8}\int 4C^2S^2\,dx = \frac{C^3S^3}{6} + \frac{1}{8}\int S_2^2\,dx$$

$$= \frac{C^3S^3}{6} + \frac{1}{16}\int (1 - C_4)\,dx = \frac{C^3S^3}{6} + \frac{x}{16} - \frac{S_4}{64} + \text{constant}$$

Remark : $\int S_m S_n\,dx$, $\int S_m C_n\,dx$, $\int C_m C_n\,dx$

Use

$$2S_mS_n = C_{m-n} - C_{m+n}$$

$$2S_mC_n = S_{m-n} + S_{m+n}$$

$$2C_mC_n = C_{m-n} + C_{m+n}$$

$$\int t^m e^n \, dx \qquad t = \tan x, \quad e = \sec x$$

$$dt = e^2 \, dx \quad \text{or} \quad de = t\,e\,dx$$

$$\int t^{2k+1} e^n \, dx = \int t^{2k} e^{n-1} \, de = \int (e^2-1)^k e^{n-1} \, de$$

$$\int t^m e^{2\ell} \, dx \overset{\ell \geq 1}{=} \int t^m e^{2\ell-2} \, dt = \int t^m (1+t^2)^{\ell-1} \, dt$$

$$\int t^{2k+1} = \int \frac{S^{2k+1}}{C^{2k+1}} \, dx = -\int \frac{(1-C^2)^k}{C^{2k+1}} \, dC$$

$m=2k$, $n=0$: $$\int t^{2k} \, dx = \int t^{2k-2} (e^2-1) \, dx = \int t^{2k-2} \, dt - \int t^{2k-2} \, dx$$

$(2k) \to (2k-2)$

$\to \cdots \to (2)$

Here $\int t^2 \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C$

$$\int t^{2k} e^{2l+1}\,dx \longrightarrow \text{integration by parts}$$

$$\underline{\int t^m e^n\,dx} \overset{m\to m-2}{=} \int t^{m-1} e^{n-1}\,de$$

$$= \frac{1}{n}\int t^{m-1}\,de^n = \frac{1}{n}\left(t^{m-1}e^n - \int e^n\,dt^{m-1}\right)$$

$$= \frac{1}{n}\left(t^{m-1}e^n - (m-1)\int t^{m-2}e^n\,e^2\,dx\right)$$

$$\overset{e^2=t^2+1}{=} \frac{1}{n}\left(t^{m-1}e^n - (m-1)\int t^m e^n\,dx - (m-1)\int t^{m-2}e^n\,dx\right)$$

$$\Rightarrow \int t^m e^n\,dx = \frac{1}{n+m-1}t^{m-1}e^n - \frac{m-1}{n+m-1}\int t^{m-2}e^n\,dx$$

(or $n\to n-2$)

$$\int t^m e^n\,dx = \frac{1}{n+m-1}t^{m+1}e^{n-2} + \frac{(n-2)}{n+m-1}\int t^m e^{n-2}\,dx$$

$$(m,n) \begin{array}{l} \nearrow (m-2,n) \\ \searrow (m,n-2) \end{array} \to \cdots \to (1,1) \text{ or } (1,0) \text{ or } (0,1)$$

$$(1,1): \int \tan x \sec x \, dx = \sec x + C$$

$$(1,0) = \int t \, dx = \int \frac{s}{c} dx = -\int \frac{dc}{c} = -\ln|\cos x| + \text{constant}$$

$$(0,1) = \int \frac{1}{c} dx = \int \frac{c}{c^2} dx = \int \frac{ds}{1-s^2} \begin{pmatrix} \text{section} \\ 8.5 \end{pmatrix}$$

$$= \frac{1}{2} \int \left(\frac{1}{s+1} - \frac{1}{s-1} \right) ds = \frac{1}{2} \ln \left| \frac{1+\sin x}{\sin x - 1} \right| + \text{const}$$

$$= \ln \sqrt{\frac{1+\sin x}{1-\sin x}} + C$$

Eg $\int \tan^4 x \, dx$

$= \int \tan^2 x (\sec^2 x - 1) \, dx$

$= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$

$= \int \tan^2 x \, d\tan x - \int (\sec^2 x - 1) \, dx$

$= \dfrac{\tan^3 x}{3} - \tan x + x + C$

Trigonometric substitution

$\int f(x)\,dx$, f contains factors

of a^2+x^2, $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$
$(a>0)$

a^2+x^2: $x = a\tan\theta$, $a^2+x^2 = a^2\sec^2\theta$, $dx = a\sec^2\theta\,d\theta$

$\sqrt{a^2-x^2}$: $x = a\sin\theta$, $a^2-x^2 = a^2\cos^2\theta$ $dx = a\cos\theta\,d\theta$

$\sqrt{x^2-a^2}$: $x = a\sec\theta$, $x^2-a^2 = a^2\tan^2\theta$ $dx = a\tan\theta\sec\theta\,d\theta$

If we choose

$$\theta = \tan^{-1}\left(\frac{x}{a}\right) \Longleftrightarrow \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \Longleftrightarrow \begin{matrix} \cos\theta > 0 \\ \sec\theta > 0 \end{matrix} \Longleftrightarrow \sqrt{a^2+x^2} = a\sec\theta.$$

$$\theta = \sin^{-1}\left(\frac{x}{a}\right) \Longleftrightarrow \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \Longleftrightarrow \cos\theta > 0 \Longleftrightarrow \sqrt{a^2-x^2} = a\cos\theta$$

$$\theta = \sec^{-1}\left(\frac{x}{a}\right) \Longleftrightarrow \theta \in \underbrace{\left(0, \frac{\pi}{2}\right)}_{\substack{\tan\theta > 0 \\ x > 0}} \cup \underbrace{\left(\frac{\pi}{2}, \pi\right)}_{\substack{\tan\theta < 0 \\ x < 0}}$$

$$\Longleftrightarrow \sqrt{x^2-a^2} = a|\tan\theta| = \begin{matrix} a\tan\theta, & x > 0 \\ -a\tan\theta, & x < 0 \end{matrix}$$

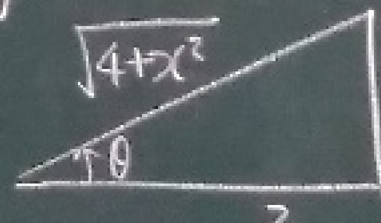
Eg: $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$ $\qquad x = 2\tan\theta \quad dx = 2\sec^2\theta\, d\theta$

$\theta = \tan^{-1}\left(\frac{x}{2}\right)$

$$= \int \frac{2\sec^2\theta\, d\theta}{2\underbrace{\sqrt{\sec^2\theta}}_{\sec\theta}} = \int \sec\theta\, d\theta$$

$$= \frac{1}{2}\ln\frac{1+\sin\theta}{1-\sin\theta} + C$$

$\tan\theta = \frac{x}{2}$



$\sin\theta = \frac{x}{\sqrt{4+x^2}}$

$$= \frac{1}{2}\ln\left(\frac{\sqrt{4+x^2}+x}{\sqrt{4+x^2}-x}\right) + C$$

$$= \frac{1}{2}\ln\left(\frac{(\sqrt{\ }+x)(\sqrt{\ }+x)}{(\sqrt{\ }-x)(\sqrt{\ }+x)}\right) + C = \ln\left(\sqrt{1+\left(\frac{x}{2}\right)^2} + \frac{x}{2}\right) + C$$