Techniques of integration

* integration by parts

$$\int f(x)\, g'(x)\, dx$$

$$\frac{d}{dx}\int f g'\, dx = f g' = (fg)' - f'g$$

$$= \frac{d}{dx}\left(fg - \int f'g\, dx\right)$$

$$\Rightarrow \int f g'\, dx = fg - \int f'g\, dx \quad (+C)$$

not needed

or $\int f\, dg = fg - \int g\, df$

Eg $\int x \cos x \, dx$

$$= \begin{cases} \int x \, d\sin x \\ \int \cos x \, d\frac{x^2}{2} \end{cases} = \begin{cases} x \sin x - \int \sin x \, dx \\ \underbrace{\frac{x^2}{2} \cos x - \int \frac{x^2}{2} \, d\cos x}_{NG} \end{cases}$$

gets more complicated

$$= x \sin x + \cos x + C$$

Check: $\frac{d}{dx}(x \sin x + \cos x)$

$$= \sin x + x \cos x - \sin x = x \cos x \quad (\text{ok})$$

$\underline{\text{Eg}}$ $\displaystyle\int \underbrace{\ln x}_{f} \, \underbrace{dx}_{dg} \quad x > 0$

$$= x \ln x - \int x \, d\ln x$$

$$= x \ln x - \int x \frac{1}{x} \, dx$$

$$= x \ln x - x + C$$

check $\frac{d}{dx}\left(x \ln x - x\right) = \ln x + x\frac{1}{x} - 1 = \ln x$

(OK)

Eg $\int x^2 e^x \, dx$

$$= \int x^2 \, de^x$$

$$= x^2 e^x - \int e^x \, dx^2$$

$$= x^2 e^x - 2\int x e^x \, dx$$

$$= x^2 e^x - 2\int x \, de^x$$

$$= x^2 e^x - 2\left(x e^x - \int e^x \, dx\right)$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

Eg $\int \cos^n x \, dx$ (n = even)

(if n = odd = $2k+1$. $\int C^{2k} C \, dx = \int (1-S^2)^k \, dS$)

$C = \cos x$. $S = \sin x$

$= \int C^{n-1} C \, dx = \int C^{n-1} dS$

$= C^{n-1} S - \int S \, dC^{n-1}$

$= C^{n-1} S - (n-1) \int S C^{n-2} (-S) \, dx$

$= C^{n-1} S + (n-1) \int C^{n-2} (1 - C^2) \, dx$

$$\therefore \int C^{n}\,dx = C^{n-1}S + (n-1)\left(\int C^{n-2}\,dx - \int C^{n}\,dx\right)$$

$$\Rightarrow n\int C^{n}\,dx = C^{n-1}S + (n-1)\int C^{n-2}\,dx$$

check: $nC^{n} = (n-1)C^{n-2}(-S^{2}) + C^{n} + (n-1)C^{n-2}$

$= (n-1)C^{n-2}(C^{2}-1) + C^{n} + (n-1)C^{n-2}$

$= nC^{n}$ OK

or $\int \cos^{n}x\,dx$

$$= \frac{1}{n}\cos^{n-1}x\,\sin x + \frac{(n-1)}{n}\int C^{n-2}\,dx$$

$$\int C^{n}\,dx \rightarrow \int C^{n-2}\,dx \rightarrow \cdots \rightarrow \int C^{2}\,dx \rightarrow \int 1\,dx$$

Alternatively. $n = 2k$

$$\int \cos^{2k} x \, dx = \int (\cos^2 x)^k \, dx$$

$$\cos 2x = 2\cos^2 x - 1$$

$$= \int \left(\frac{1 + \cos 2x}{2} \right)^k dx$$

$$= \int (C_0 + C_1 \cos 2x + \cdots + C_j \cos^j 2x + \cdots + C_k \cos^k 2x) \, dx$$

Reduces faster, result is more complicated

Definite integrals

$$\int f g' \, dx = fg - \int g f' \, dx$$

$$\Rightarrow \int (f g' + f' g) \, dx = fg$$

$$fg = \text{Antiderivative of } \int f g' + g f' \, dx$$

$$\overset{\text{FTCII}}{\Longrightarrow} fg \Big|_a^b = \int_a^b (f g' + g f') \, dx$$

$$\text{i.e. } \int_{x=a}^{b} f \, dg = fg \Big|_a^b - \int_{x=a}^{b} g \, df$$

FTCII = Fundamental Theorem of Calculus, Part II

Eg $\int_0^4 x e^x \, dx = \int_{x=0}^4 x \, de^x$

$= xe^x \Big|_0^4 - \int_0^4 e^x \, dx$

$= xe^x \Big|_0^4 - e^x \Big|_0^4$

$= (x-1)e^x \Big|_0^4 = 3e^4 + e^0$

$= 3e^4 + 1$

Trigonometric integrals

$$\int \cos^m x \sin^n x \, dx$$

if $m=2k+1$ $= \int C^{2k} S^n C \, dx = \int (1-S^2)^k S^n \, dS$

if $n=2l+1$ $= \int C^m S^{2l} S \, dx = -\int C^m (1-C^2)^l \, dC$

if $\begin{matrix} m=2k \\ n=2l \end{matrix}$ $= \int \left(\frac{1+C_2}{2}\right)^k \left(\frac{1-C_2}{2}\right)^l dx \qquad C_2 = \cos 2x$

(Method 1)

$$= \int a_0 + a_1 C_2 + \cdots + a_{k+l} C_2^{k+l} \, dx$$

Reduce from $C^{2k} S^{2l}$ to C_2^j, $0 \le j \le k+l$

Method 2 $\quad m=2k,\ n=2l$ (assume $k>1$ or $l>1$)

(if $k=1,\ l=1$, apply $C_2=2C^2-1=1-2S^2$ directly)

Assume $k>1$

$$\int C^{2k}S^{2l}\,dx=\int C^{2k-1}S^{2l}C\,dx=\int C^{2k-1}S^{2l}\,dS$$

$$=\frac{1}{2l+1}\int C^{2k-1}\,dS^{2l+1}=\frac{1}{2l+1}\left(C^{2k-1}S^{2l+1}-\int S^{2l+1}\,dC^{2k-1}\right)$$

$$\int S^{2l+1}\,dC^{2k-1}=(2k-1)\int S^{2l+1}C^{2k-2}(-S)\,dx$$

$$=(1-2k)\int S^{2l+2}C^{2k-2}\,dx \underset{S^2=1-C^2}{=} (1-2k)\int\left(S^{2l}C^{2k-2}-S^{2l}C^{2k}\right)dx$$

$$\int C^{2k} S^{2\ell}\,dx$$

$$= \frac{1}{2\ell+1}\left(C^{2k-1} S^{2\ell+1} + (2k-1)\left(\int C^{2k-2} S^{2\ell}\,dx - \int C^{2k} S^{2\ell}\,dx \right) \right)$$

$$\left(1 + \frac{2k-1}{2\ell+1}\right) \int C^{2k} S^{2\ell}\,dx$$

$$= \frac{1}{2\ell+1} C^{2k-1} S^{2\ell+1} + \frac{2k-1}{2\ell+1} \int C^{2k-2} S^{2\ell}\,dx$$

Reduce from $\int C^{2k} S^{2\ell}\,dx$ to $\int C^{2k-2} S^{2\ell}\,dx$