Hyperbolic functions (skip inverse hyperbolic functions)

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \quad \left(\begin{array}{c}\text{hyperbolic} \\ \text{sine}\end{array}\right)$$

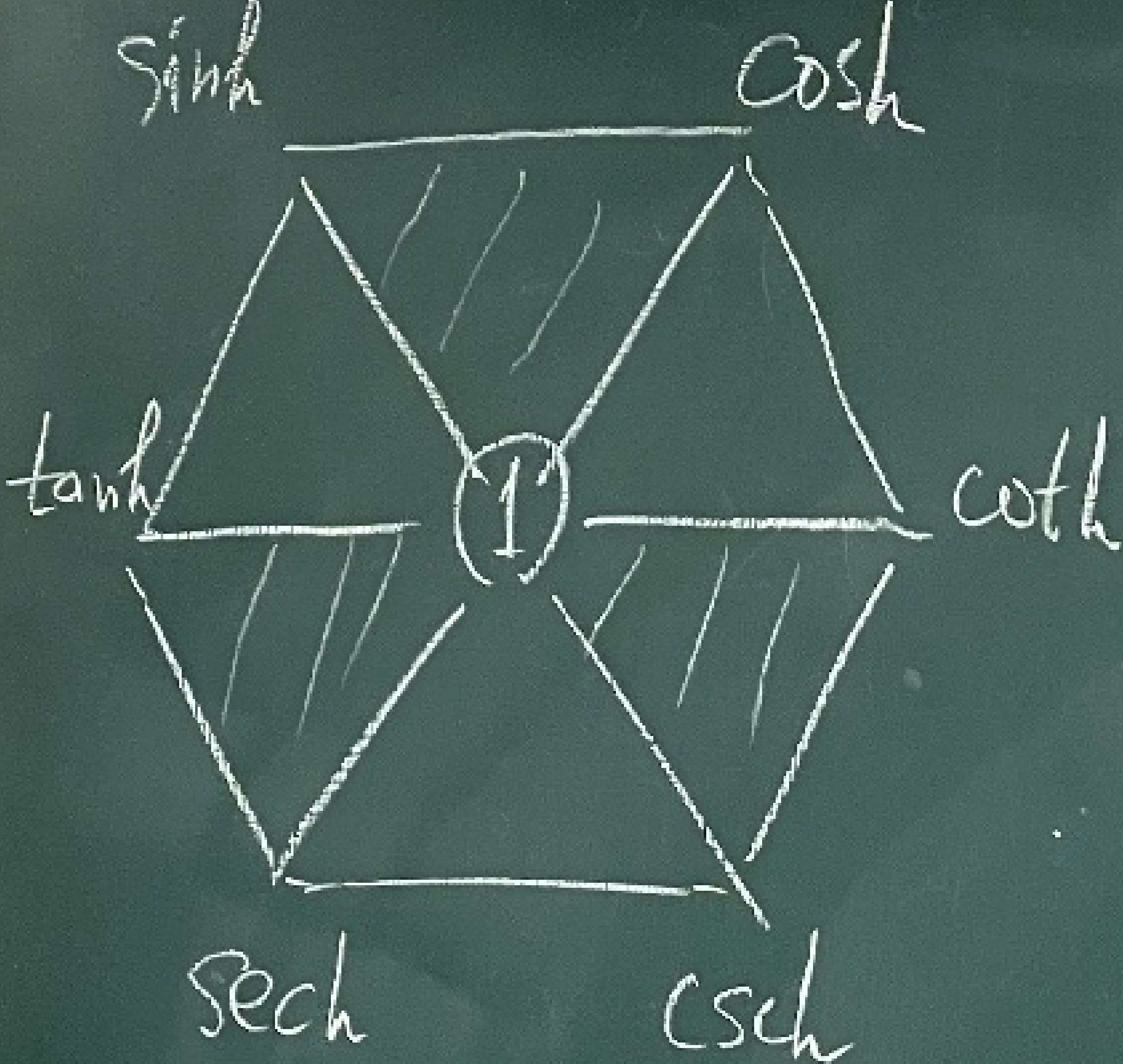
$$\cosh x = \frac{e^{x} + e^{-x}}{2} \quad \left(\begin{array}{c}\text{hyperbolic} \\ \text{cosine}\end{array}\right)$$

$$\tanh x = \sinh x / \cosh x$$

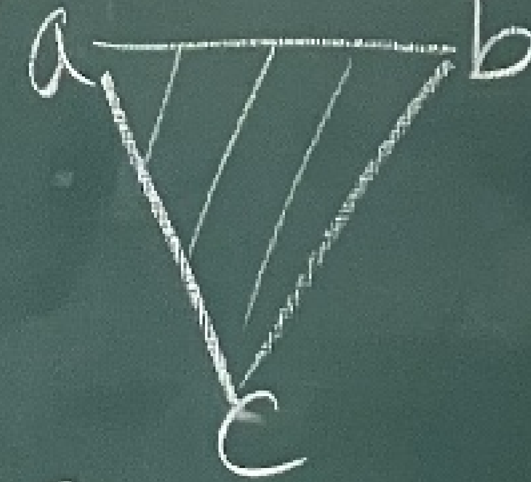
$$\coth x = \cosh x / \sinh x \quad (x \neq 0)$$

$$\operatorname{sech} x = 1 / \cosh x$$

$$\operatorname{csch} x = 1 / \sinh x \quad (x \neq 0)$$



(i)



$$b^2 - a^2 = c^2$$

(ii) diagonal entries are reciprocal to each other.

(iii) Any corner entry is the product of two neighboring corner entries

Example in (i)

(*) $\cosh^2 x = \dfrac{e^{2x} + 2 + e^{-2x}}{4}$

$\sinh^2 x = \dfrac{e^{2x} - 2 + e^{-2x}}{4}$

$\Rightarrow \cosh^2 x - \sinh^2 x = 1$

(*) $1 - \tanh^2 x = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x}$

Similarly for $\coth^2 x - 1 \qquad = \operatorname{sech}^2 x$

Other identities:

$$\sinh 2x = \frac{e^{2x}-e^{-2x}}{2} = 2\,\frac{e^{x}+e^{-x}}{2}\,\frac{e^{x}-e^{-x}}{2}$$

$$= 2\sinh x \cosh x$$

$$\cosh 2x = \frac{e^{2x}+e^{-2x}}{2}$$

$$= \frac{1}{2}\left(\left(e^{x}+e^{-x}\right)^2-2\right) = \frac{1}{2}\left(\left(e^{x}-e^{-x}\right)^2+2\right)$$

$$= 2\cosh^2 x - 1 \qquad = 2\sinh^2 x + 1$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

Derivative of hyperbolic functions

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\coth x = \frac{d}{dx}\frac{\cosh x}{\sinh x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = \frac{d}{dx}(\cosh x)^{-1} = \frac{-\sinh x}{\cosh^2 x} = -\tanh x \operatorname{sech} x$$

$$\frac{d}{dx}\operatorname{csch} x = \frac{d}{dx}(\sinh x)^{-1} = \frac{-\cosh x}{\sinh^2 x} = -\coth x \operatorname{csch} x$$

integrals of hyperbolic functions

Table 7.6

Eg: $\frac{d}{dx} \tanh\sqrt{1+x^2}$

$= \left(\operatorname{sech}^2\sqrt{1+x^2}\right) \frac{x}{\sqrt{1+x^2}}$

Eg: $\int_0^1 \tanh x \, dx = \int_0^1 \frac{\sinh x}{\cosh x} dx$

$= \int_0^1 \frac{d\cosh x}{\cosh x} = \ln\cosh x \Big|_0^1$

$= \ln\cosh 1$

Remark: $\int_0^1 \tanh x \, dx$

Let $u = \cosh x$ $\quad du = \sinh x \, dx$

$$= \int_{x=0}^{1} \frac{du}{u} = \int_{u=1}^{\cosh 1} \frac{du}{u} = \ln u \Big|_1^{\cosh 1} = \ln \cosh 1$$

Eg: $\int \sinh^2 x \, dx$

$$= \int \frac{e^{2x} - 2 + e^{-2x}}{4} dx = \int \frac{\cosh 2x - 1}{2} dx$$

$$= \frac{1}{4} \sinh 2x - \frac{x}{2} + C$$

Eg: $\int_0^{\ln 2} 4e^x \sinh x \, dx$

$= \int_0^{\ln 2} 2(e^{2x} - 1) \, dx$

$= \left(e^{2x} - 2x\right)\Big|_0^{\ln 2}$

$= 3 - 2\ln 2$

Def rate of growth

If $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = \begin{cases} \infty \\ L, & 0<L<\infty \\ 0, \end{cases}$

we say

$f(x)$ grows $\begin{matrix} \text{faster than} \\ \text{at the same rate as} \\ \text{slower than} \end{matrix}$ $g(x)$

as $x \to \infty$

Eg $-5x^5 + 7x^4 - 2x^2 - 1$

grows at the same rate as x^5

as $x \to \infty$

Eg: Among $f(x) = e^{0.01x}$

$g(x) = x^7$, $h(x) = (\ln x)^{1000}$

$f(x)$ grows fastest, $h(x)$ grows slowest

as $x \to \infty$

Since $\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0$, $\lim_{x \to \infty} \left(\frac{h(x)}{g(x)}\right)^{\frac{1}{1000}} = 0$

Def: $f(x) = o(g(x))$ as $x \to \infty$, 0, 0^+

$(f \text{ "}<\text{" } g)$

if $\lim\limits_{\substack{x \to \infty \\ 0 \\ 0^+}} \frac{f(x)}{g(x)} = 0$

Def: $f(x) = O(g(x))$ as $x \to \infty$, 0, 0^+

$(f \text{ "}\leq\text{" } g$ in some sense$)$

if there exists $M>0$, $N>0$ $(\delta>0)$

such that

$$\left|\frac{f(x)}{g(x)}\right| \leq M \quad \text{for all} \quad \begin{array}{l} x > N \\ x \in (-\delta, \delta) \\ x \in (0, \delta) \end{array}$$

or simply write:

if $\left|\frac{f(x)}{g(x)}\right| \leq M$ as $x \to \infty$, 0, 0^+

Remark

(i) $\lim \left|\frac{f}{g}\right| = 0 \iff f = o(g)$

(ii) $\lim \left|\frac{f}{g}\right| = L \implies f = O(g)$

$\qquad\qquad\qquad\quad \not\Longleftarrow \; g = O(f)$

(iii) $\lim \left|\frac{f}{g}\right| = \begin{matrix}0\\ L\end{matrix} \;\; \begin{matrix}\Longrightarrow \\ \not\Longleftarrow\end{matrix} \;\; f = O(g)$

Eg: $f(x) = x \qquad g(x) = x(2+\sin x)$

$\implies f = O(g) \qquad g = O(f) \qquad x \to \infty$

But $\lim |f/g|$ does not exist

(iv) $f = o(g) \Rightarrow f = O(g)$

$\not\Leftarrow$

Eg: $(\ln x)^{1000} = o(x^7)$

$x^7 = o(e^{0.001x})$ as $x \to \infty$

Eg: $(\ln x)^{1000} = o(x^{-2})$

$x^{-2} = o\left(e^{\frac{1}{x^2}}\right)$ as $x \to 0^+$

Eg: $x^p = o(x^q)$ as $x \to \infty$ if $p < q$

Eg: $x + \sin x = O(x)$ as $x \to \infty$

Eg: $e^x + x^3 = O(e^x)$ as $x \to \infty$