Linear differential equations

$$\frac{dy}{dx} = \underbrace{-p(x)y + q(x)}_{\text{linear in } y}$$

(textbook $p, q \to P, Q$)

$$\frac{dy}{dx} + p(x)y = q(x)$$

Multiply $e^{P(x)}$ on both sides. $P'(x) = p(x)$

$$e^{P(x)} y'(x) + e^{P(x)} P'(x) y(x) = e^{P(x)} q(x)$$

$$\left(e^{P(x)} y(x)\right)' = e^{P(x)} q(x)$$

$$e^{P(x)} y(x) = \int e^{P(x)} g(x)\, dx$$

$$\therefore\ y(x) = e^{-P(x)} \underbrace{\int e^{P(x)} g(x)\, dx}_{\text{contains an integral constant to be determined}}$$

Special case:

$$p(x) = \frac{k}{x}$$

$$P(x) = k \ln|x| \quad (\text{may assume} = k \ln x)$$

Multiply $e^{P(x)} = e^{k \ln x} = x^k$ on both sides

$$\text{i.e. } (x^k y)' = x^k y' + k x^{k-1} y = x^k f(x)$$

Eg

$$\begin{cases} x\dfrac{dy}{dx} = x^2 + 3y & x>0 \\ y(1) = 1 \end{cases}$$

$$\underbrace{xy' - 3y}_{\textcircled{1}} = x^2 \text{ ——— } (*)$$

Since $(x^{-3}y)' = \underbrace{x^{-3}y' - 3x^{-4}y}_{\textcircled{2}}$

Compare ① with ②

$\Rightarrow$ Multiply x^{-4} on both sides of $(*)$

$$(x^{-3}y)' = x^{-2}, \Rightarrow x^{-3}y = -x^{-1} + C$$

$$y(1) = 1 \Rightarrow C = 2, \Rightarrow y = -x^2 + 2x^3$$

Rm $\begin{cases} (x^{-3}y)' = x^{-2} \\ y(1) = 1 \end{cases}$

$$\Rightarrow x^{-3}y(x) - 1^{-3}y(1) = \int_1^x t^{-2}\,dt = -t^{-1}\Big|_1^x$$

Eg $y' + 2y = 3,\ y(0) = 1$

$$\Rightarrow e^{2x}y' + 2e^{2x}y = 3e^{2x}$$

$$(e^{2x}y)' = 3e^{2x}$$

$$e^{2x}y(x) - e^{2\cdot 0}\cdot y(0) = \int_0^x 3e^{2t}\,dt,\ y = -\frac{1}{2}e^{-2x} + \frac{3}{2}$$

Eg: Solve $e^{x} y' + 2e^{2x} y = 1$

Solve $y' + 2e^{x} y = e^{-x}$

$$p(x) = 2e^{x}, \quad P(x) = 2e^{x}$$

Multiply $e^{2e^{x}}$ on both sides

$$\left(e^{2e^{x}} y\right)' = e^{2e^{x} - x}$$

$$y(x) = e^{-2e^{x}} \int e^{2e^{x} - x}\, dx = e^{-2e^{x}} \int^{x} e^{2e^{t} - t}\, dt$$

$$\left(\neq \int e^{-x}\, dx\right)$$

Eg $\frac{dy}{dx} + xy = x$, $y(0) = 2$

$p(x) = x$, $P(x) = \frac{x^2}{2}$

$$\Rightarrow \left(e^{\frac{x^2}{2}} y\right)' = x e^{\frac{x^2}{2}} = \left(e^{\frac{x^2}{2}}\right)'$$

$$\Rightarrow \left(e^{\frac{t^2}{2}} y(t) \Big|_0^x = \left(e^{\frac{t^2}{2}} \right) \Big|_0^x \right)$$

or $e^{\frac{x^2}{2}} y(x) = e^{\frac{x^2}{2}} + C$

$y(0) = 2 \Rightarrow C = 1$ $\therefore$ $y(x) = 1 + e^{\frac{-x^2}{2}}$

Eg $(1+x)y' + y = \sqrt{x}$

Sol $y' + \frac{1}{1+x} y = \frac{\sqrt{x}}{1+x}$

$p(x) = \frac{1}{1+x}$. $P = \ln(1+x)$

Multiply by $e^{\ln(1+x)} = (1+x)$

$\Rightarrow \left((1+x)y\right)' = \sqrt{x} = x^{\frac{1}{2}}$

$\therefore y(x) = \frac{1}{1+x}\left(\frac{2}{3}x^{\frac{3}{2}} + C\right)$