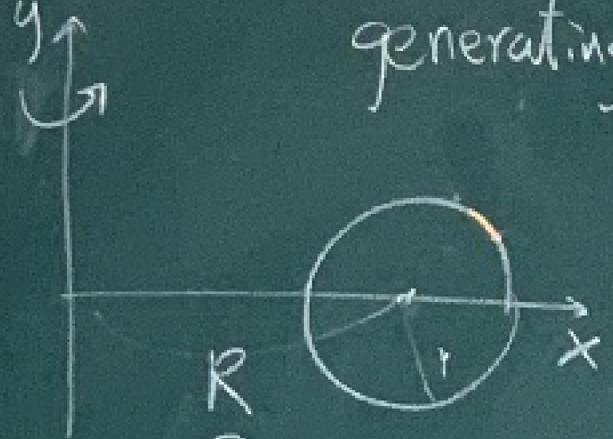
Surface area of a donut



generating curve $(x-R)^2 + y^2 = r^2$

$$y = \pm\sqrt{r^2-(x-R)^2}$$

$$y' = \mp \frac{x-R}{\sqrt{r^2-(x-R)^2}}$$

$$A^{K.F} = \int_{x=R-r}^{R+r} 2\pi r \, ds$$

"$ds = \sqrt{dx^2 + dy^2}$"

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i = \sqrt{1+\left(\frac{\Delta x_i}{\Delta y_i}\right)^2}\,\Delta y_i$$

$$A = 2\int_{R-r}^{R+r} 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\int_{R-r}^{R+r} 2\pi x \frac{r}{\sqrt{r^2-(x-R)^2}}\,dx$$

$$= 2\int_{R-r}^{R+r} 2\pi(x-R+R)\frac{r}{\sqrt{r^2-(x-R)^2}}\,d(x-R)$$

$$\mathbb{X} = x-R$$

$$= 2\int_{\mathbb{X}=-r}^{r} 2\pi(\mathbb{X}+R)\frac{r}{\sqrt{r^2-\mathbb{X}^2}}\,d\mathbb{X}$$

$$= 4\pi r\left(\int_{-r}^{r}\frac{\mathbb{X}}{\sqrt{r^2-\mathbb{X}^2}}\,d\mathbb{X} + \int_{-r}^{r}\frac{R}{\sqrt{r^2-\mathbb{X}^2}}\,d\mathbb{X}\right)$$

$$= 4\pi r\left(\frac{-1}{2}\int_{-r}^{r}\frac{d(r^2-\mathbb{X}^2)}{\sqrt{r^2-\mathbb{X}^2}} + R\int_{-r}^{r}\frac{1}{\sqrt{1-\left(\frac{\mathbb{X}}{r}\right)^2}}\,d\frac{\mathbb{X}}{r}\right)$$

$$= 4\pi r\left(0 + R\sin^{-1}\left(\frac{\mathbb{X}}{r}\right)\Big|_{\mathbb{X}=-r}^{r}\right)$$

$$= 4\pi^2 rR$$

Separable differential equations

First order differential equations

Solve $y(x)$ from $\frac{dy(x)}{dx} = F(x, y(x))$

Special cases.

(I): separable: $F(x,y) = g(x)H(y)$

(II): linear: $F(x,y) = -P(x)y + Q(x)$

Eg: (Separable) $\frac{dy}{dx} = (1+y)e^x, \; y > -1$

<u>Sol</u>: formally

$$\frac{dy}{1+y} = e^x dx$$

$$\int \frac{dy}{1+y} = \int e^x dx$$

$$\ln|1+y| = e^x + C$$

$$\overset{\underset{\shortparallel}{y>-1}}{\ln(1+y)} = e^x + C$$

$$y = \exp(e^x + C) - 1$$

(C to be determined by additional boundary condition)

In reality

$$\frac{1}{1+y}\frac{dy}{dx} = e^x$$

$$\int \frac{1}{1+y}\frac{dy}{dx}dx = \int e^x dx$$

$$\shortparallel$$

$$\int \frac{1}{1+y} dy$$

Eg: Solve $y(x+1)\frac{dy}{dx} = x(y^2+1)$

Sol: $\int \frac{y}{1+y^2}\,dy = \int \frac{x}{x+1}\,dx$

$$\frac{1}{2}\int \frac{dy^2}{1+y^2} = \int \left(1 - \frac{1}{x+1}\right)dx$$

$$\frac{1}{2}\int \frac{d(1+y^2)}{1+y^2} = x - \int \frac{1}{x+1}\,d(x+1)$$

$$\frac{1}{2}\ln(1+y^2) = x - \ln|1+x| + C$$

$$\ln(1+y^2) = 2x - \ln(1+x)^2 + 2C$$

$$y = \pm\sqrt{e^{2(x+c)}/(1+x)^2 - 1}$$