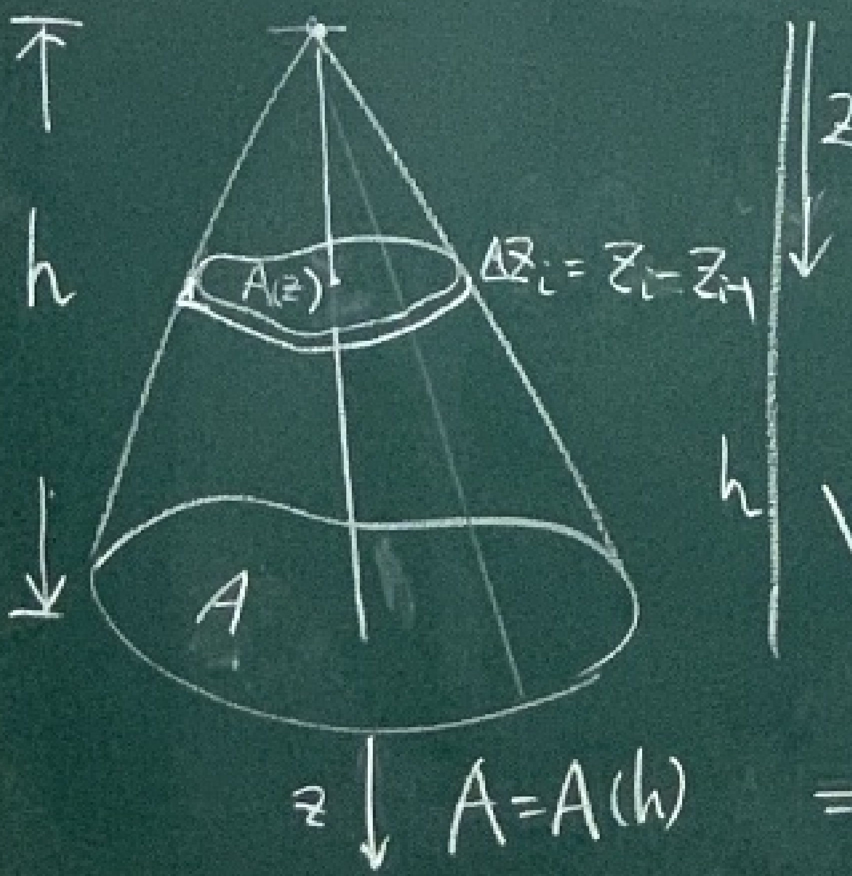
Volume by Cross-section

Eg: Volume of Cones



$A = A(h)$

$$\frac{A(z)}{A(h)} = \left(\frac{z}{h}\right)^2$$

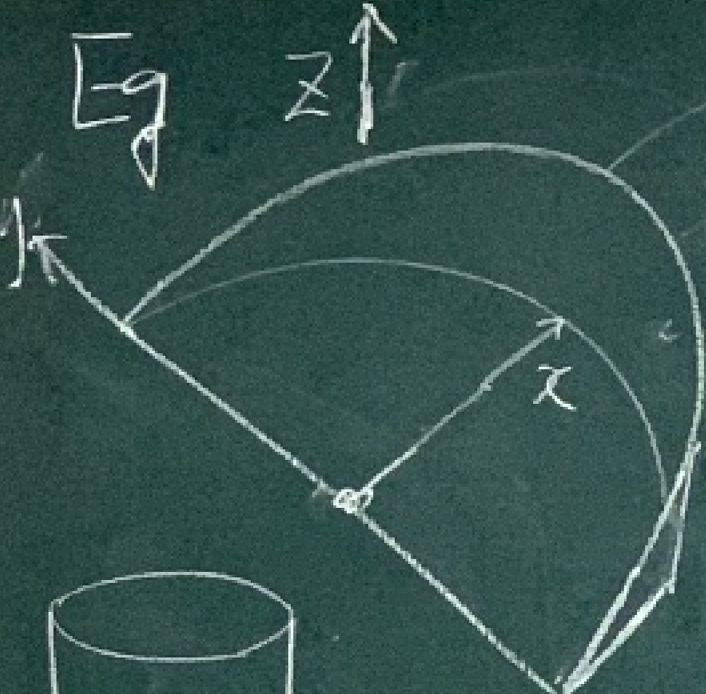
$$A(z) = \frac{A}{h^2} z^2$$

$$\Delta V = A \Delta z$$

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(z_i) \Delta z_i$$

$$= \int_0^h A(z)\, dz = \frac{Ah}{3}$$

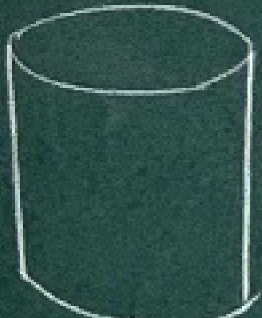
Eg



$x = z$ (plane)

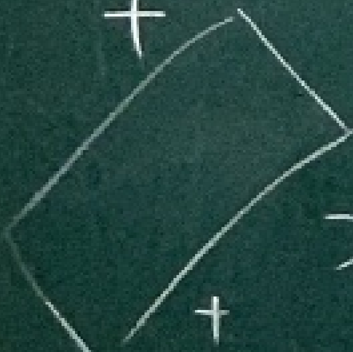
$x^2 + y^2 = 9$ (Cylinder)

(I): $\int_0^3 A(x)\,dx$



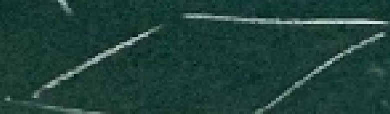
$x^2+y^2=9$

$+$



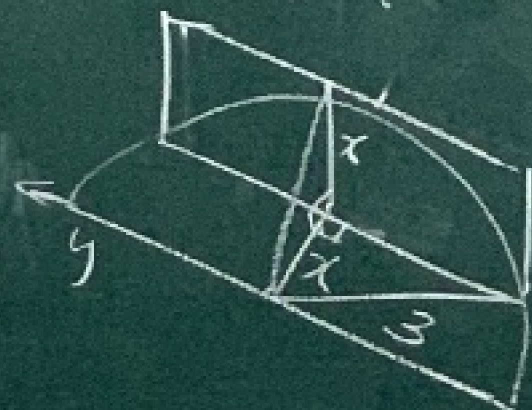
$x = z$

$+$



$x=0$

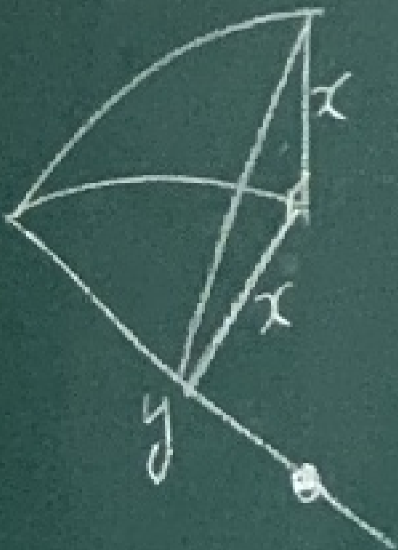
$$A(x) = \underbrace{\left(2\sqrt{9-x^2}\right)}_{底} \underbrace{x}_{高}$$



$$V = \int_0^3 A(x)\,dx$$

$= 18$

(II) $V = \int_{-3}^{3} \tilde{A}(y)\,dy$



$$\tilde{A}(y) = \frac{x^2}{2} = \frac{9-y^2}{2}$$

$$V = \int_{-3}^{3} \frac{9-y^2}{2}\,dy$$

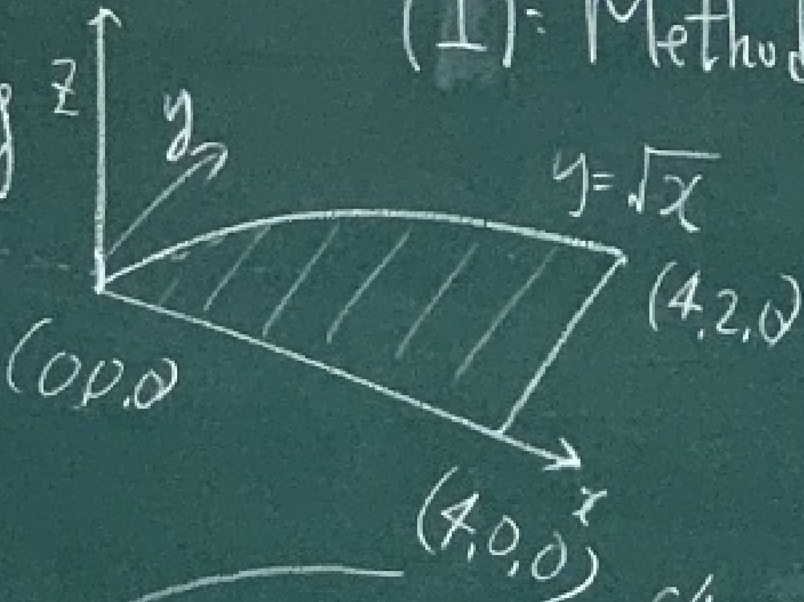
$$= \frac{9y}{2} - \frac{1}{6}y^3 \Big|_{-3}^{3}$$

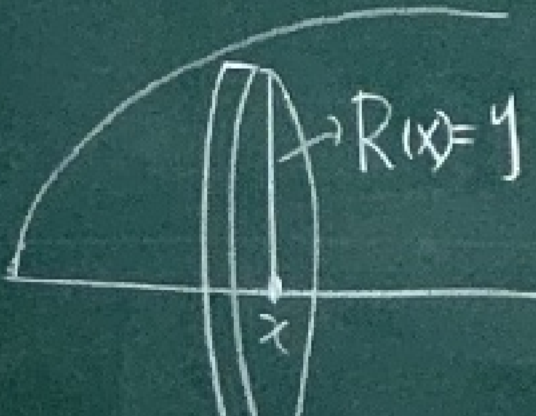
$$= 18$$

Volume of Revolution

(I): Method of disks

Eg





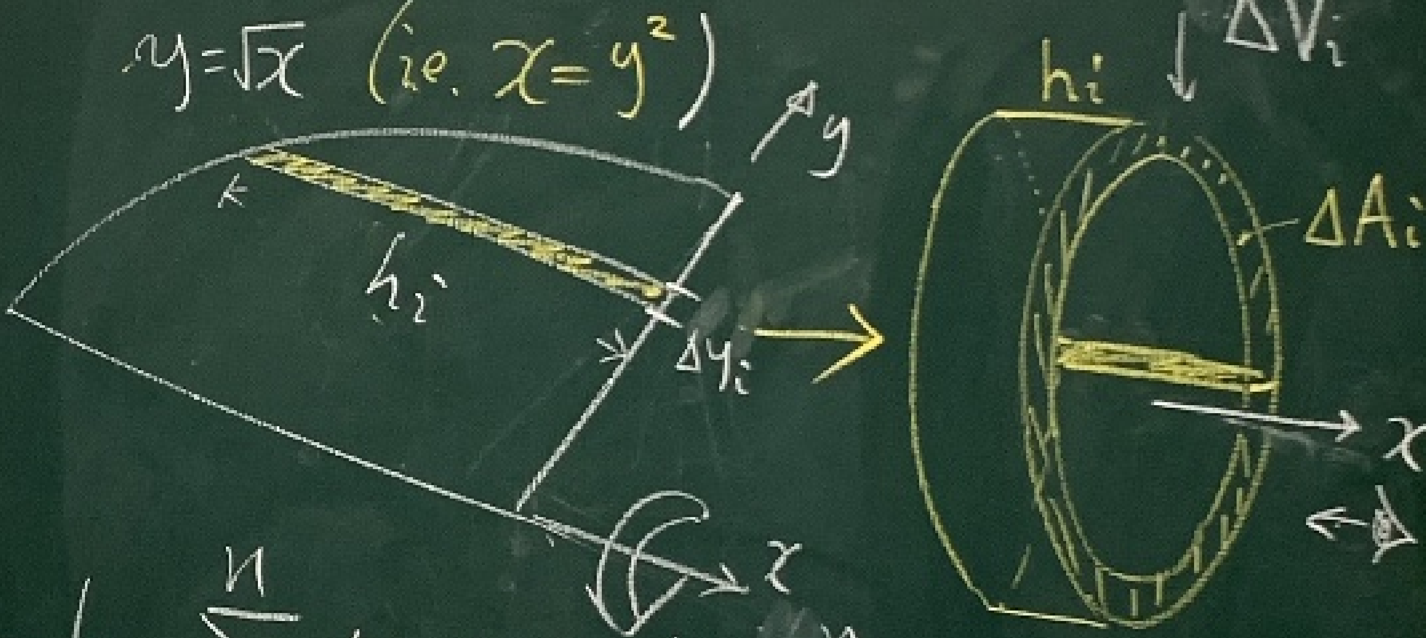
$$V=\int_0^4 \pi(\sqrt{x})^2\,dx$$

$$=8\pi$$

Volume of Revolution

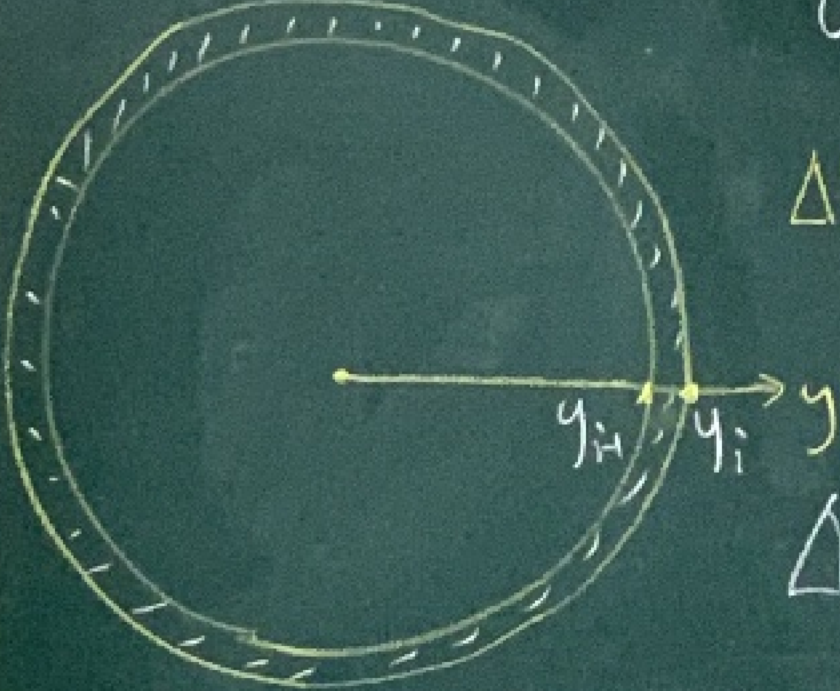
(II): Cylindrical Shells

$y=\sqrt{x}$ (i.e. $x=y^2$)



$$V=\lim_{n\to\infty}\sum_{i=1}^{n}\Delta V_i=\lim_{n\to\infty}\sum_{i=1}^{n}\Delta A_i\cdot h_i$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}2\pi y_i\, h_i\, \Delta y_i$$



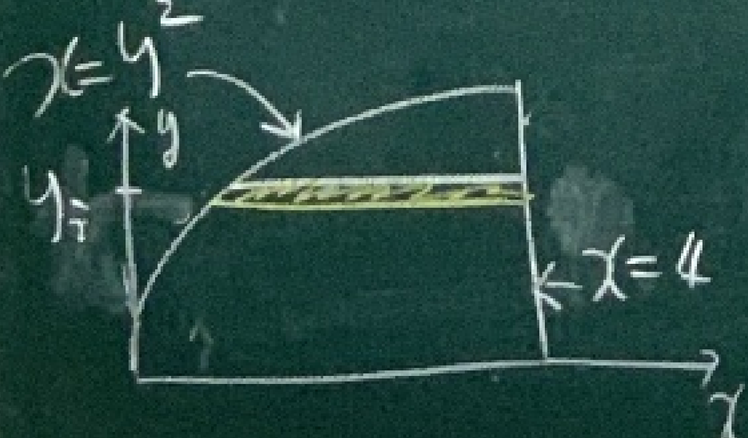
Outer radius

inner radius

$$\Delta y_i = y_i - y_{i-1}$$

$$\Delta A_i \simeq 2\pi y_i \, \Delta y_i$$

$$h_i = 4 - y_i^2, \quad V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi y_i \, \Delta y_i \left(4 - y_i^2\right)$$

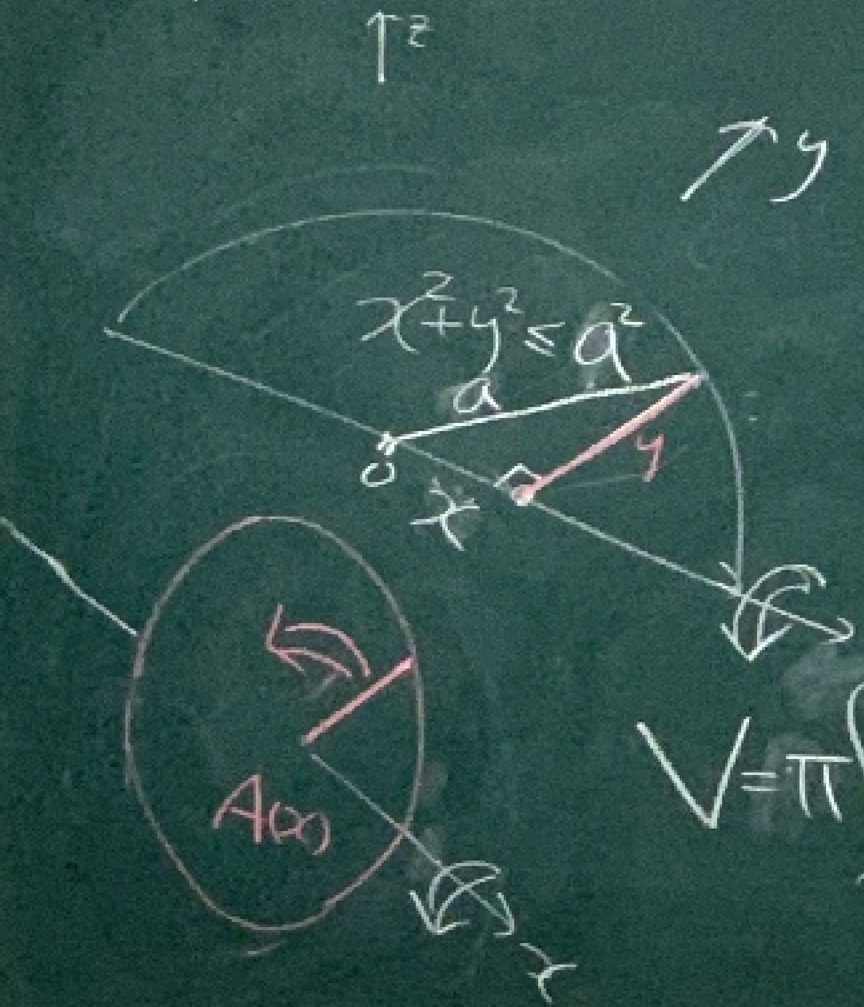


$$= \int_0^2 2\pi y \left(4 - y^2\right) dy$$

$$= 8\pi$$

Eg: Volume of ball (radius $= a$)

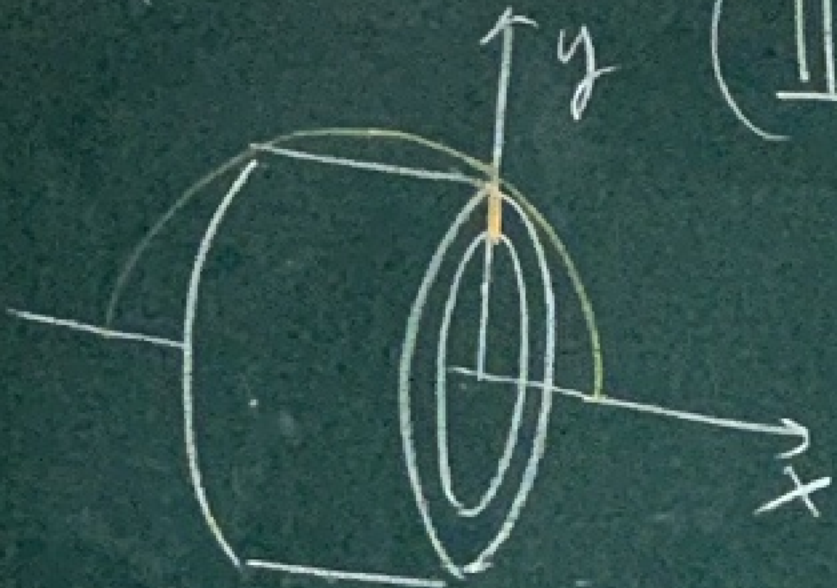
(I): disks

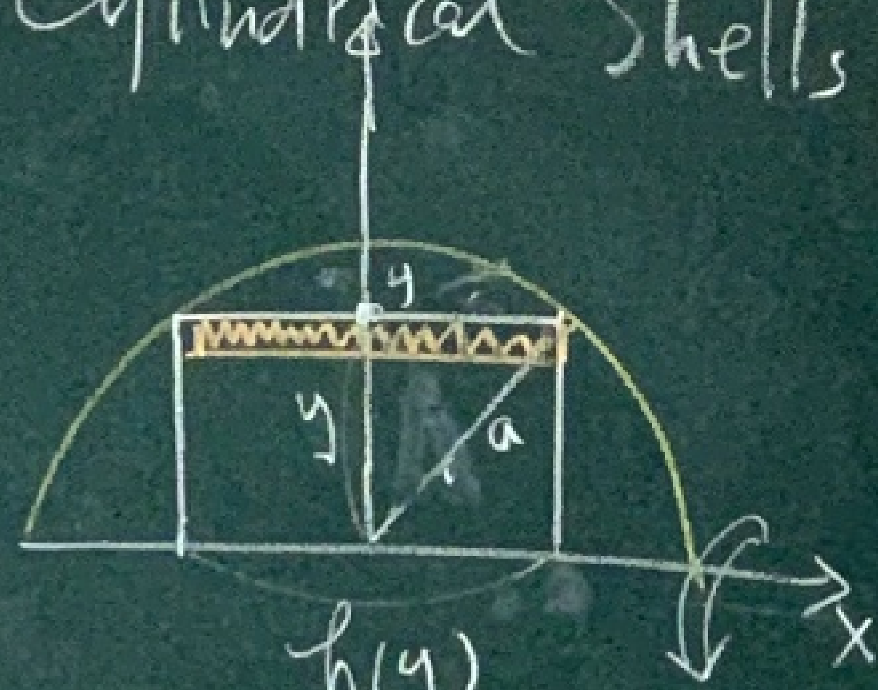


$$A(x) = \pi y^2$$
$$= \pi(a^2 - x^2)$$

$$V = \pi\int_{-a}^{a}(a^2 - x^2)\,dx = \frac{4\pi a^3}{3}$$

(II) Cylindrical Shells





$$h(y) = 2\sqrt{a^2 - y^2}$$

Volume of ball

$$= \int_0^a 2\pi y \, 2\sqrt{a^2-y^2}\, dy = \frac{4\pi a^3}{3}$$