Fundamental Theorem of Calculus

Part I: If f is cont. on $[a,b]$ then $F(x) \stackrel{\text{def}}{=\!=} \int_a^x f(t)\,dt$ is cont. on $[a,b]$ and diff. on (a,b) and

$$F'(x) = f(x)$$

i.e

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Part II: If G is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

i.e.

$$G(b) - G(a) = \int_a^b G'(x)\,dx$$

Note notation change $F \leftrightarrow G$ from textbook

Pf: (Part I)

$$\frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0}\frac{\int_a^{x+h} * - \int_a^x *}{h}$$

$$= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}$$

$= ($average of f on $[x, x+h]) \in [m, M]$

Thm 3. $\lim_{h\to 0} f(c)$, $c\in(x, x+h)$ or $(x+h, x)$

$h\to 0, c\to x$ $= f(x)$

pf (Part II)

Let $F(x)=\int_a^x f(t)\,dt$

$$\xRightarrow{\text{Part I}} F'(x) = f(x) = G'(x)$$

$$\Rightarrow (F-G)' = 0 \quad \therefore\ G(x) = F(x) + C$$

$$G(b)-G(a) = \big(F(b)+C\big) - \big(F(a)+C\big)$$

$$= F(b) - F(a) = \int_a^b f(t)\,dt - \underbrace{\cancel{\int_a^a}}_{=0}$$

Examples (a) $\frac{d}{dx}\int_a^x (t^3+1)\,dt = x^3+1$

(b) $\frac{d}{dx}\int_x^0 \sin(t^2)\,dt = \frac{d}{dx}\left(-\int_0^x \right) = -\sin(x^2)$

(c) $\frac{d}{dx}\int_1^{x^2} e^{\frac{1}{t}}\,dt = \frac{d}{dx}F(x^2)$

$= \frac{d}{du}F(u)\Big|_{u=x^2} \cdot \frac{dx^2}{dx} = e^{\frac{1}{x^2}} \cdot 2x$

(d) $\frac{d}{dx}\int_{x^2}^{x^3} \frac{1}{t^2+1}\,dt$

$$= \frac{d}{dx}\left(\int_1^{x^3} - \int_1^{x^2}\right)$$

$$= \frac{3x^2}{(x^3)^2+1} - \frac{2x}{(x^2)^2+1}$$

Examples (Part II)

(a) $\int_0^{\pi} \cos x\,dx = \sin x\Big|_0^{\pi} = \sin\pi - \sin 0$

$= 0$

(b) $\int_1^4 \frac{3}{2}\sqrt{x} - \frac{4}{x^2}\,dx$

$= \int_1^4 \frac{3}{2}x^{\frac{1}{2}} - 4x^{-2}\,dx = x^{\frac{3}{2}} + 4x^{-1}\Big|_1^4 = 4$

(c) $\int_0^1 \frac{dx}{x^2+1} = \tan^{-1} x\Big|_0^1 = \frac{\pi}{4}$

(d) $\int_{-1}^1 \sqrt{1-x^2}\,dx =$  $= \frac{\pi}{2}$

Eg $\int_0^1 (x^3+x)^5(3x^2+1)\,dx$

$$= \frac{1}{6}\int_0^1 \frac{d}{dx}(x^3+x)^6\,dx$$

$$= \frac{1}{6}(x^3+x)^6 \Big|_0^1$$

$$= \frac{64-0}{6} = \frac{32}{3}$$

In general (Reverse engineering of the Chain Rule)

$$\int f(g(x))\, g'(x)\, dx = \int \frac{d}{dx} F(g(x))\, dx$$

where $F'(u) = f(u)$

$$= F(g(x)) + C$$

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(x)) \Big|_{x=a}^{x=b}$$

Eg: $\int_0^1 \sqrt{2x+1}\, dx$

$$= \frac{1}{2}\int_0^1 \sqrt{2x+1}\; \underbrace{2}_{=\frac{d}{dx}(2x+1)}\, dx$$

$$= \frac{1}{2}\int_{x=0}^{x=1} u(x)^{\frac{1}{2}}\, u'(x)\, dx \qquad u(x) = 2x+1$$

$$\left(= \frac{1}{2}\int_{u=1}^{u=3} u^{\frac{1}{2}}\, du = \frac{1}{3}u^{\frac{3}{2}}\Big|_{u=1}^{3}\right)$$

$$= \int_{x=0}^{1} \frac{1}{3}\frac{d}{dx}(2x+1)^{\frac{3}{2}}\, dx = \frac{1}{3}(2x+1)^{\frac{3}{2}}\Big|_{x=0}^{1}$$

$$\int f(\sin x)\cos x\,dx = F(\sin x) + C$$

$$\int f(\cos x)\sin x\,dx = -F(\cos x) + C$$

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$$\int e^{g(x)} g'(x)\,dx = \int e^{g(x)}\,dg(x) = e^{g(x)} + C$$

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C$$

Eg $\int \sec^2(7x+3)\,dx = \frac{\tan(7x+3)}{7} + C$

$$\int_0^1 x^2 e^{x^3}\,dx = \frac{e^{x^3}}{3}\Big|_0^1 = \frac{e-1}{3}$$

$$\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x\,dx}{e^{2x}+1} = \int \frac{de^x}{(e^x)^2+1} = \tan^{-1}(e^x)+C$$

$$\parallel$$

$$\int \frac{e^{-x}\,dx}{1+e^{-2x}} = -\int \frac{de^{-x}}{1+(e^{-x})^2} = -\tan^{-1}(e^{-x})+C$$

$$\left(\because \tan^{-1}\left(\tfrac{b}{a}\right)+\tan^{-1}\left(\tfrac{a}{b}\right)=\tfrac{\pi}{2}\right) = -\left(\tfrac{\pi}{2}-\tan^{-1}(e^x)\right)+C$$