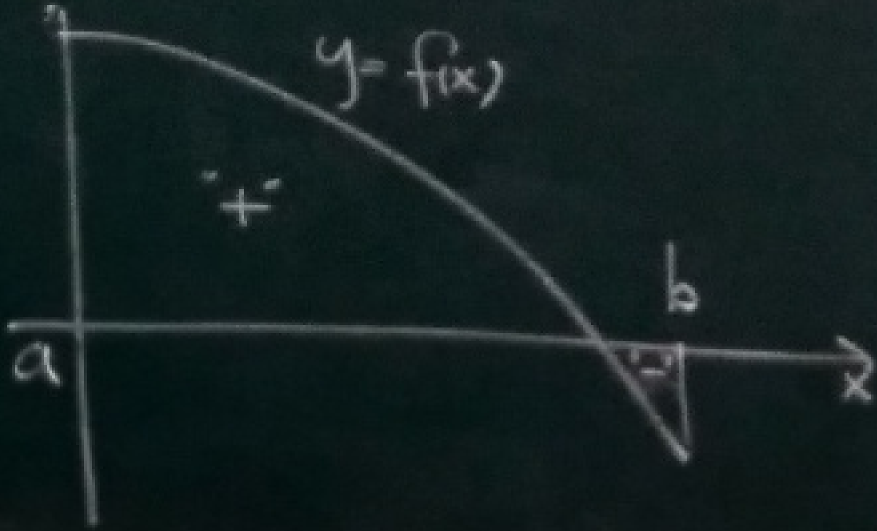
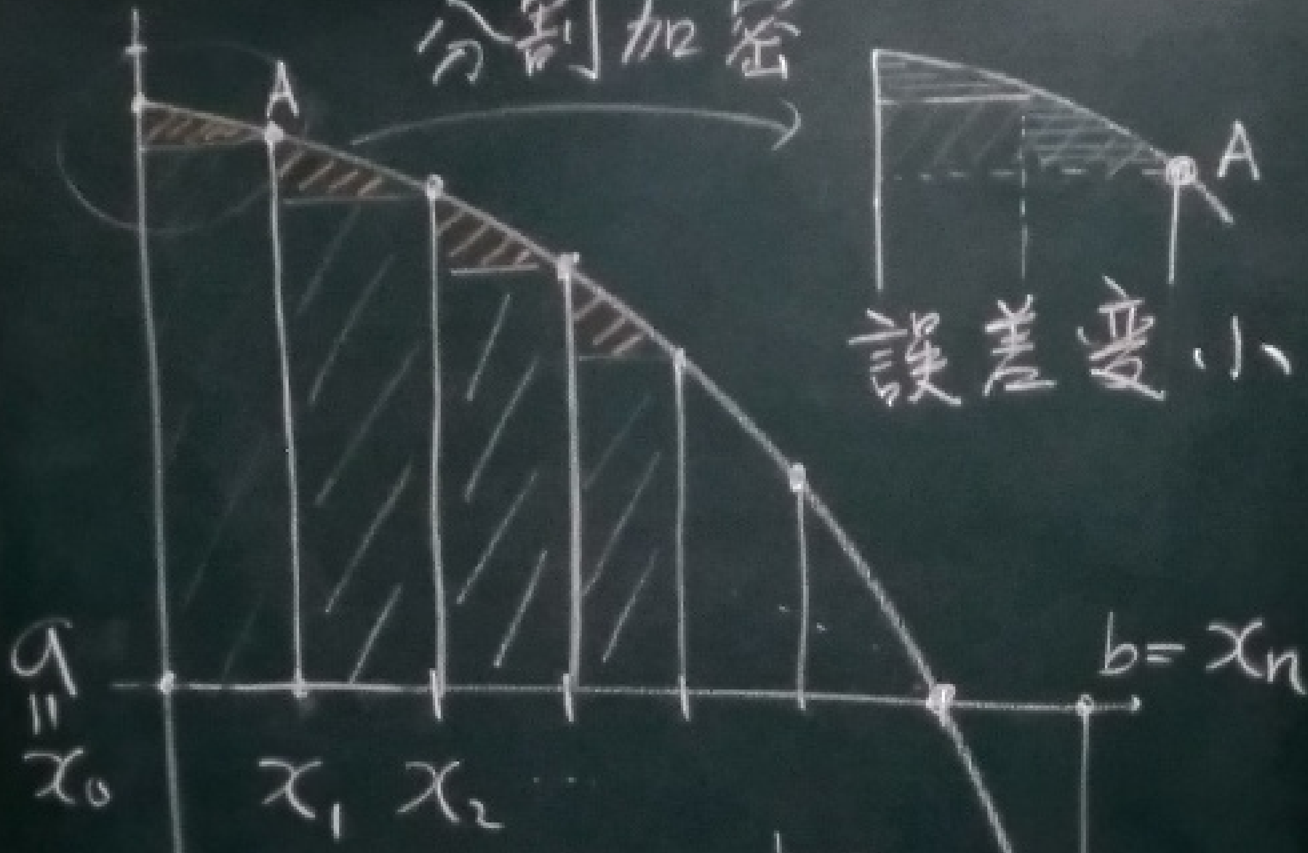
Integration.

How to compute the "signed" area under the graph of $f(x)$ on $[a, b]$





$$x_i - x_{i-1} = \frac{b-a}{n}$$

$$\text{Area} \cong \frac{b-a}{n} f(x_1) + \frac{b-a}{n} f(x_2) + \cdots + \frac{b-a}{n} f(x_n)$$

$$a = x_0 < x_1 < \cdots < x_n = b$$

$$x_i - x_{i-1} = \frac{b-a}{n}$$

Approximate area

$$A_n = \sum_{k=1}^{n} (x_k - x_{k-1}) f(x_k)$$

Signed area can be computed by $\lim_{n\to\infty} A_n$

We denote the limit as $\int_a^b f(x)\,dx$
if it exists

Definite integral

$$a = x_0 < x_1 \cdots < x_n = b$$

$$P \stackrel{\text{def}}{=} \{x_0, \cdots\cdots x_n\}$$

$$\|P\| \stackrel{\text{def}}{=} \max_{1 \le k \le n} (x_k - x_{k-1})$$

$$\int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{\|P\| \to 0} \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k)$$

where $c_k \in [x_{k-1}, x_k]$

Why use $\lim_{\|P\| \to 0}$ instead of $\lim_{n \to \infty}$?

$$f(x) = \begin{cases} 0 & x = \text{rational} \\ 1 & x = \text{irrational} \end{cases}$$

Note: $\lim_{\|P\| \to 0}$ does not exist for this f(x)!

$\lim_{n \to \infty} A_n = 0$ on $[0, 1]$

$\because x_k = \dfrac{k}{n}$ $f(x_k) = 0$

$\lim_{n \to \infty} A_n = \underline{\sqrt{2} - 1}$ on $[0, \sqrt{2} - 1] \subseteq [0, 1]$

$\because x_k = \frac{k}{n}(\sqrt{2} - 1)$. $f(x_k) = 1$

$\Rightarrow \lim_{n \to \infty} A_n$ is not a good definition

Def $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k) = I$:

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|P\| < \delta \implies \left| \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k) - I \right| < \varepsilon$$

Thm If f is cont. (or only finitely many jump discont.) on $[a, b]$, then this limit exists.

Rm ① $\int_b^a f(x)dx = -\int_a^b f(x)dx$

Table 5.6

② $\int_a^a f(x)dx = 0$ ....

Rm If $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k)$

exists (we say f is integrable on [a,b])

(The limit $\int_a^b f(x)dx$ is the definite integral of f on [a,b]

then it equals $\lim_{n \to \infty} A_n$

Eg: $\int_0^1 x^2\,dx$

$\frac{1}{n}$ $\quad \underset{0}{\overset{x_0}{\cdot}} \quad \underset{\frac{1}{n}}{\overset{x_1}{\cdot}} \quad \underset{\frac{2}{n}}{\overset{x_2}{\cdot}} \quad \cdots \quad \underset{1}{\overset{x_n}{\cdot}}$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right), \quad x_k = \frac{k}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{k^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^2}{n^3} = \lim_{n\to\infty} \frac{\frac{n(n+1)(2n+1)}{6}}{n^3} = \frac{1}{3}$$

Rm: If f is integrable, then for any a, b, c

$$\int_a^c f(x)\,dx = \int_a^b + \int_b^c \quad (\text{don't need } a<b<c)$$