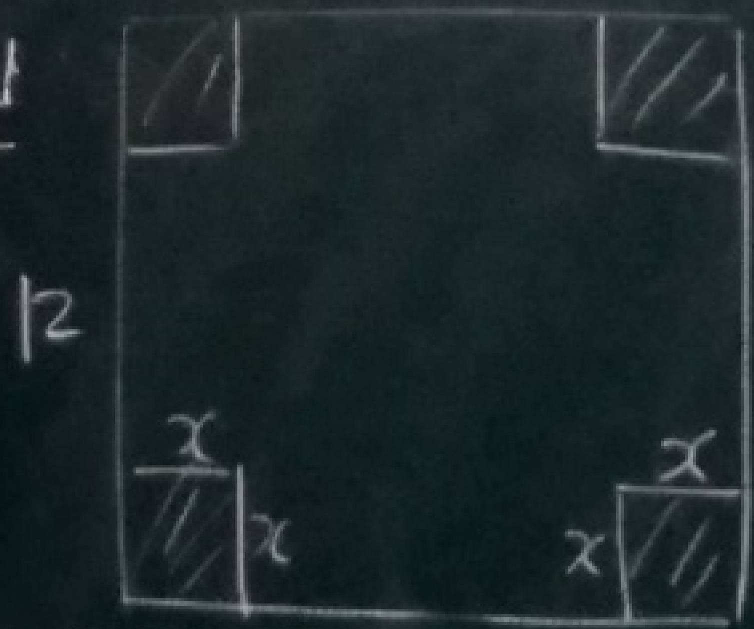
Applied Optimization

Eg:



Find maximal volume of the box

Sol:

$$V(x) = (12-2x)^2 x = 4x^3 - 48x^2 + 144x$$

$$V'(x) = 12x^2 - 96x + 144 = 12(x^2 - 8x + 12) = 12(x-2)(x-6)$$

$V'(x)$		$+$		$-$		$+$
	0		2		6	
V		$\nearrow$		$\searrow$		$\nearrow$

$\Rightarrow \begin{cases} \text{local max } x=2 \\ \text{local min: } x=0,6 \end{cases}$

$x=2 \Rightarrow$ only local max $\Rightarrow$ abs. max

Eg2.

Fix $V=\pi r^2 h = 1000$

find minimal surface area of the cylinder

Sol: $A(r) = 2\pi r^2 + 2\pi r h$

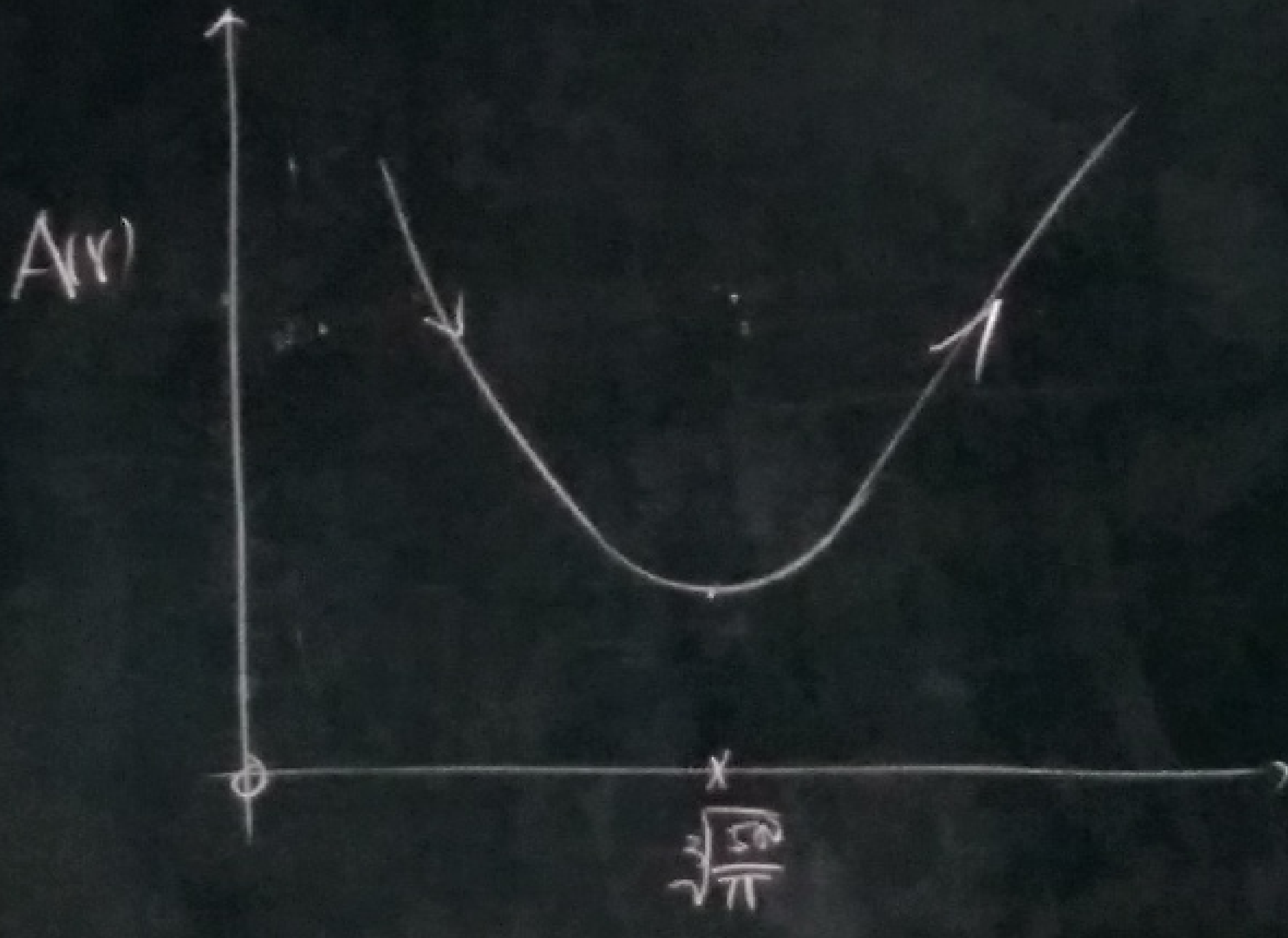
$$= 2\pi r^2 + 2\pi \cdot \frac{1000}{\pi r}$$

$$= 2\left(\pi r^2 + \frac{1000}{r}\right), \quad r>0$$

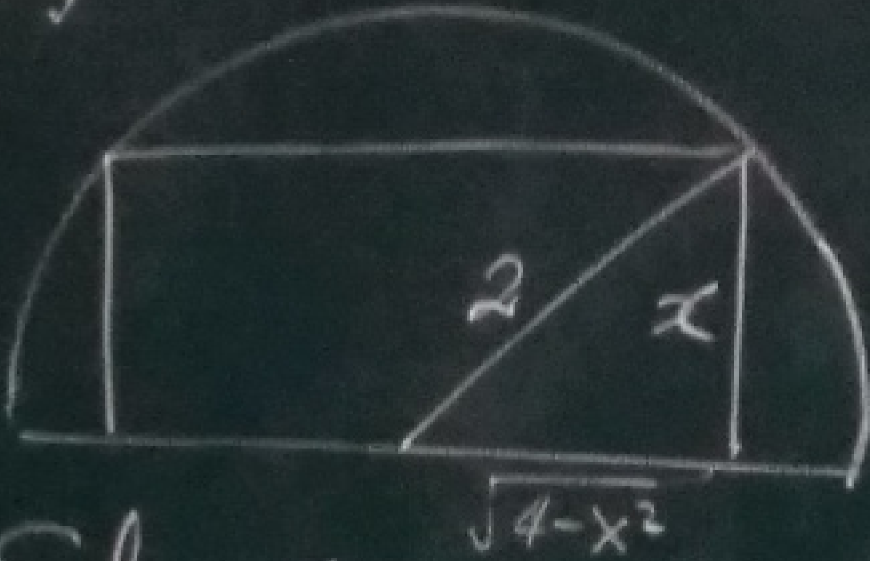
$$A'(r) = 4\pi r - 2000/r^2$$

$$= \frac{4\pi\left(r^3 - \frac{500}{\pi}\right)}{r^2} \Rightarrow (*) \begin{cases} A'(r) > 0, & r > \sqrt[3]{\frac{500}{\pi}} \\ \quad = 0 & \quad = \\ \quad < & \quad < \end{cases}$$

$(*)$ $A\left(\sqrt[3]{\frac{500}{\pi}}\right) =$ abs. min


A(r)
0
$\sqrt[3]{\frac{50}{\pi}}$

Eg 3

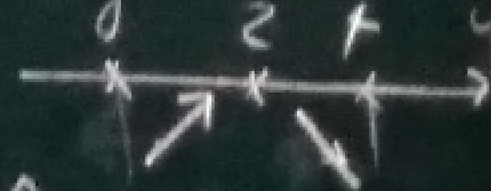


maximize area of rectangle

Sol: $A(x) = 2\sqrt{4-x^2} \cdot x$ (Method 1 textbook)

Method 2: $\max \left(\frac{A}{2}\right)^2 = B = (4-x^2)x^2$ on $0 \leq x \leq 2$

$= \max\limits_{0 \leq y \leq 4} (4-y)y$, $\frac{dB}{dy} = 2(2-y)$



$y = 2 \Rightarrow x = \sqrt{2}$; $A(\sqrt{2})$ = only local max

= abs max

__Eg__ x : # of products, $x \geq 0$

$r(x) = 9x$: revenue

$c(x) = x^3 - 6x^2 + 15x$: cost

Find maximal profit ($p = r - c$)

__Sol.__ $p'(x) = -3x^2 + 12x - 6$

$= -3(x^2 - 4x + 2)$

$= -3(x - x_-)(x - x_+)$

where $x_\pm = 2 \pm \sqrt{2}$

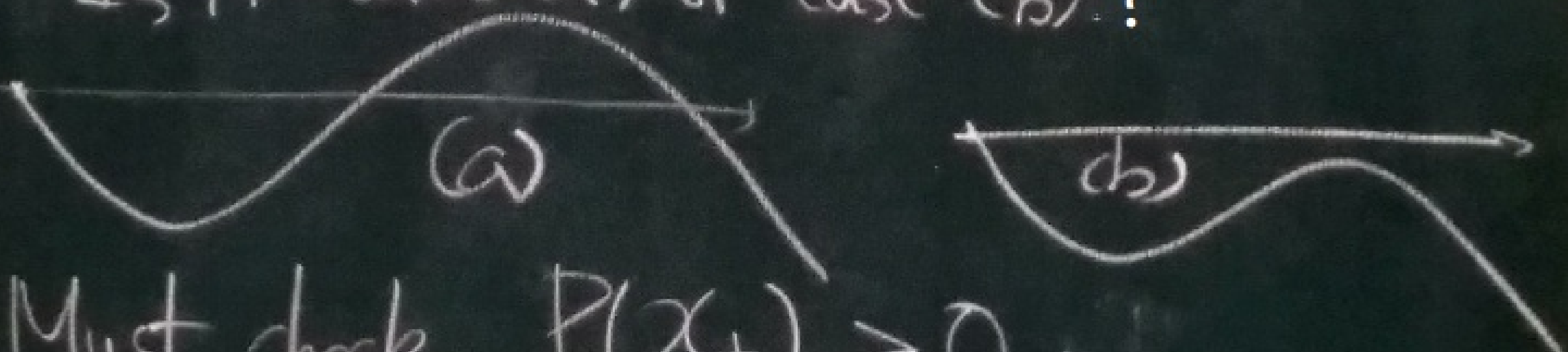
$p'(x)$		$-$		$+$		$-$
x	0		x_-		x_+	
		$\searrow$		$\nearrow$		$\searrow$

$P(x_-)$ = local min

there is no abs min

$P(0)$, $P(x_+)$: local max

Is it case (a) or case (b)?



Must check $P(x_+) > 0$

or $p(x) = -x(x^2 - 6x + 6)$ has 3 roots $(6^2 - 4 \cdot 6 > 0)$ $\Rightarrow$ case (a)

$P(x_+)$ = abs. max

Antiderivative

<u>Def</u>: $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$

<u>Rm</u> If $F(x)$ is an antiderivative of $f(x)$, then so is $F(x)+C$ for any constant C.

<u>Def</u>: $\int f(x)\,dx$ = the class of all $F(x)+C$ such that $F'(x) = f(x)$

Eg $f(x)$ $\quad \int f(x)\,dx$

$$x^n \quad \begin{cases} \dfrac{x^{n+1}}{n+1}+C & n\neq -1 \\ \ln|x|+C & n=-1 \end{cases}$$

$$\sin kx \quad \frac{-1}{k}\cos kx$$

$$\cos kx \quad \frac{1}{k}\sin kx$$

$$\sec^2 kx \quad \frac{1}{k}\tan kx$$

$$\cdots\cdots$$

$$e^{kx} \quad \frac{1}{k}e^{kx}+C$$

$$a^{kx}=e^{k\ln a\,x} \quad \frac{1}{k\ln a}a^{kx}+C$$

Eg

Solve $\begin{cases} \dfrac{dv}{dt} = -32 \\ v(0) = 12 \end{cases}$

Ans: $\begin{cases} v(t) = -32t + C \\ v(0) = 12 \end{cases}$

$\Rightarrow C = 12 \quad \therefore v(t) = -32t + 12$

Eg

$$\begin{cases} \dfrac{d^2x}{dt^2} = -9.8 \\ x(0) = 10 \\ x'(0) = 0 \end{cases}$$

Sol

$$\frac{d^2x}{dt^2} = -9.8 \Rightarrow \frac{dx}{dt} = -9.8t + C_1$$

$$\Rightarrow x(t) = -4.9t^2 + C_1 t + C_2$$

$x(0) = 10 \Rightarrow C_2 = 10$; $x'(0) = 0 \Rightarrow C_1 = 0$

$$\therefore x(t) = -49t^2 + 10$$

Exercise: What if $x(0) = 10$, $x(1) = 0$?