Eg $\lim_{x\to 0} \frac{x - \sin x}{x^3}$ "$\frac{0}{0}$"

$= \lim_{x\to 0} \frac{1-\cos x}{3x^2}$ "$\frac{0}{0}$"

$= \lim_{x\to 0} \frac{\sin x}{6x}$ "$\frac{0}{0}$"

$= \lim_{x\to 0} \frac{\cos x}{6}$ not "$\frac{0}{0}$"

$= 1/6$

Eg $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x+x^2}$ "$\dfrac{0}{0}$"

$= \lim\limits_{x\to 0} \dfrac{\sin x}{1+2x} = \dfrac{0}{1}$ $\left(\text{not } \dfrac{0}{0}\right)$

Stop here

$\neq \lim\limits_{x\to 0} \dfrac{\cos x}{2} = \dfrac{1}{2}$

$\frac{\infty}{\infty}$ Version of l'Hôpital's Rule

Change "$f(a) = 0 = g(a)$"

to "$\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$"

l'Hôpital's Rule also applies

(as well as $\lim_{x\to a^{\pm}}$ and $\lim_{x\to\pm\infty}$)

"$0 \cdot \infty$ case" $\left(\triangleq \frac{0}{0} \text{ or } \frac{\infty}{\infty} \right)$

Eg $\lim_{x \to \infty} x \sin \frac{1}{x}$ "$0 \cdot \infty$"

$= \lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}}$ "$\frac{0}{0}$"

$= \lim_{x \to \infty} = \lim_{x \to \infty} \frac{\frac{-1}{x^2} \cos \frac{1}{x}}{\frac{-1}{x^2}} = 1$

Eg $\lim_{x\to\infty} \frac{\ln x}{x^{\frac{1}{2}}}$ "$\frac{\infty}{\infty}$"

$= \lim_{x\to\infty} \frac{x^{-1}}{\frac{1}{2}x^{\frac{-1}{2}}} = \lim_{x\to\infty} 2x^{\frac{-1}{2}} = 0$

Same result for $x^{\frac{1}{2}} \longrightarrow x^{k}$, $k>0$

Rm: $\boxed{e^{ax} \gg x^{b} \gg (\ln x)^{c} \text{ (as } x \to \infty) \quad \forall a,b,c>0}$

Eg

$$\lim_{x\to 0^+} \frac{e^{\frac{-1}{x}}}{x} \left(\frac{0}{0}\right) = \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x}}} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{e^{\frac{-1}{x}}}{x^2} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0^+} \frac{e^{\frac{-1}{x}}}{2x^3} \left(\frac{0}{0}\right)$$

$$= \ldots\ldots$$

$$= \lim_{x\to 0^+} \frac{\cancel{-x^{-2}}}{\cancel{-x^{-2}}\, e^{\frac{1}{x}}} \left(\frac{1}{\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{1}{e^{\frac{1}{x}}} = 0$$

Eg $\lim_{x \to 0} \frac{x^2 \cos \frac{1}{x}}{\sin x} = \lim_{x \to 0} \left(\frac{x}{\sin x}\right)\left(x \cos \frac{1}{x}\right)$

$$\stackrel{?}{=} \lim_{x \to 0} \frac{2x \cos \frac{1}{x} + \sin \frac{1}{x}}{\cos x} \qquad = 0$$

does not exist $\therefore$ "=" does not hold

Note: l'Hospital Rule only applies to RHS $= \begin{cases} L \\ +\infty \\ -\infty \end{cases}$, but not this case

Other applications "$\infty - \infty$", "1^{∞}", "0^0", "∞^0"

Eg $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$ $\quad \infty - \infty$

$= \lim_{x\to 0} \frac{x - \sin x}{x \sin x}$ $\quad \frac{0}{0}$

$= \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x}$ $\quad \frac{0}{0}$

$= \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0$

Eg: $\lim_{x\to\infty} x^{\frac{1}{x}}$ $\quad \infty^0$

$$= \lim_{x\to\infty} \left(e^{\ln x}\right)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln x}{x}} \quad \left(\frac{\infty}{\infty}\right)$$

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0$$

$$\therefore \text{Ans} = e^0 = 1$$

Rm when $x \to 0^+$, $a, b, c > 0$

$$(|\ln x|)^a \ll x^{-b} \ll e^{\frac{c}{x}}$$

Eg: $\lim_{x \to 0^+} x^x$ $\quad 0^0$

$= \lim_{x \to 0^+} e^{x \ln x}$ $\quad e^{0 \cdot \infty}$

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} (-x) = 0$$

Rm: this is the proof of the "0/0" version. It does not apply to the "infinity/infinity" version of l'Hopital's rule.

proof of l'Hôpital's Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{(f(x)-f(a))}{(g(x)-g(a))}$$

$$= \lim_{x\to a}\frac{\left(\dfrac{f(x)-f(a)}{x-a}\right)}{\left(\dfrac{g(x)-g(a)}{x-a}\right)} \underset{?}{\overset{M.V.T}{=}} \lim_{x\to a}\frac{f'(c_1)}{g'(c_2)}$$

Note: M.V.T. may give different c for f and g

Standard M.V.T. cannot be used in the proof of l'Hopital'e rule. Need the following

Cauchy's Mean Value Thm

If f, g are cont. on $[a,b]$ and diff on (a,b).

Then $\exists\, c \in (a,b)$ such that

$$\begin{vmatrix} f(b)-f(a) & f'(c) \\ g(b)-g(a) & g'(c) \end{vmatrix} = 0$$

$\underline{\underline{Pf}}$: Define $F(x) = \begin{vmatrix} f(b)-f(a) & f(x)-f(a) \\ g(b)-g(a) & g(x)-g(a) \end{vmatrix}$

$\Rightarrow F(b) = 0 = F(a) \overset{\text{Rolle's}}{\Longrightarrow} \exists\, c \in (a,b),\ F'(c) = 0$

Back to the proof of L'Hôpital's Rule

$$\frac{f(x)}{g(x)} \underset{\uparrow}{=} \frac{f(x)-f(a)}{g(x)-g(a)} \underset{\uparrow}{=} \frac{f'(c)}{g'(c)}$$

(Since $f(a) = 0 = g(a)$) Cauchy's M.V.T.

(for some c between x and a)

Take $\lim\limits_{x \to a}$ on both sides

(note that, $c \to a$ when $x \to a$)

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c \to a} \frac{f'(c)}{g'(c)} \quad \text{if RHS} = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$$

It is inconclusive if RHS is oscillatory.

This concludes the proof of lim_{x->a} "0/0" version of L'Hopital's rule. The proof of the "infinity/infinity" version is beyond the scope of this course.