Concavity and curve sketching

Recall: $f' \underset{\leq}{\geq} 0$ on $(a,b) \Rightarrow f$ is $\underset{\text{dec.}}{\text{inc.}}$ on (a,b)

Similarly,

$f'' \underset{<}{>} 0 \Rightarrow f'$ is $\underset{\text{dec.}}{\text{inc.}}$ concave $\underset{\text{down.}}{\text{up.}}$

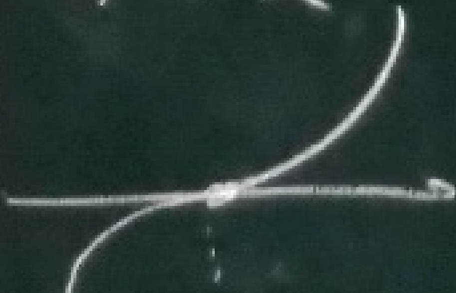
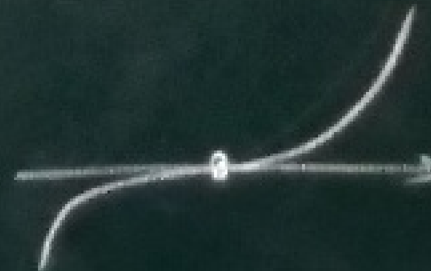
Def: $(c, f(c))$ is called a point of inflection if f changes concavity across c

ie f is concave $\underset{\text{downward}}{\text{upward}}$ on $\underset{(c, c+\delta)}{(c-\delta, c)}$ $\left(\underset{\text{or } (c-\delta, c)}{(c, c+\delta)}\right)$

downward

Examples

$f(x)=x^{\frac{5}{3}}$ $\qquad x^4 \qquad x^3$

	$f(x)=x^{\frac{5}{3}}$	x^4	x^3
f'':	$-\ \times\ +$	$+\ 0\ +$	$-\ 0\ +$
infl.?	Yes	No	Yes

$f''(c)=0 \;\not\Longrightarrow\; (c, f(c))$ is point of inflection

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infl. $\downarrow$ point of inflection?

Example: Sketch the graph of

$$f(x) = x^4 - 4x^3 + 10$$

and mark all local extremos and points of inflection, if any.

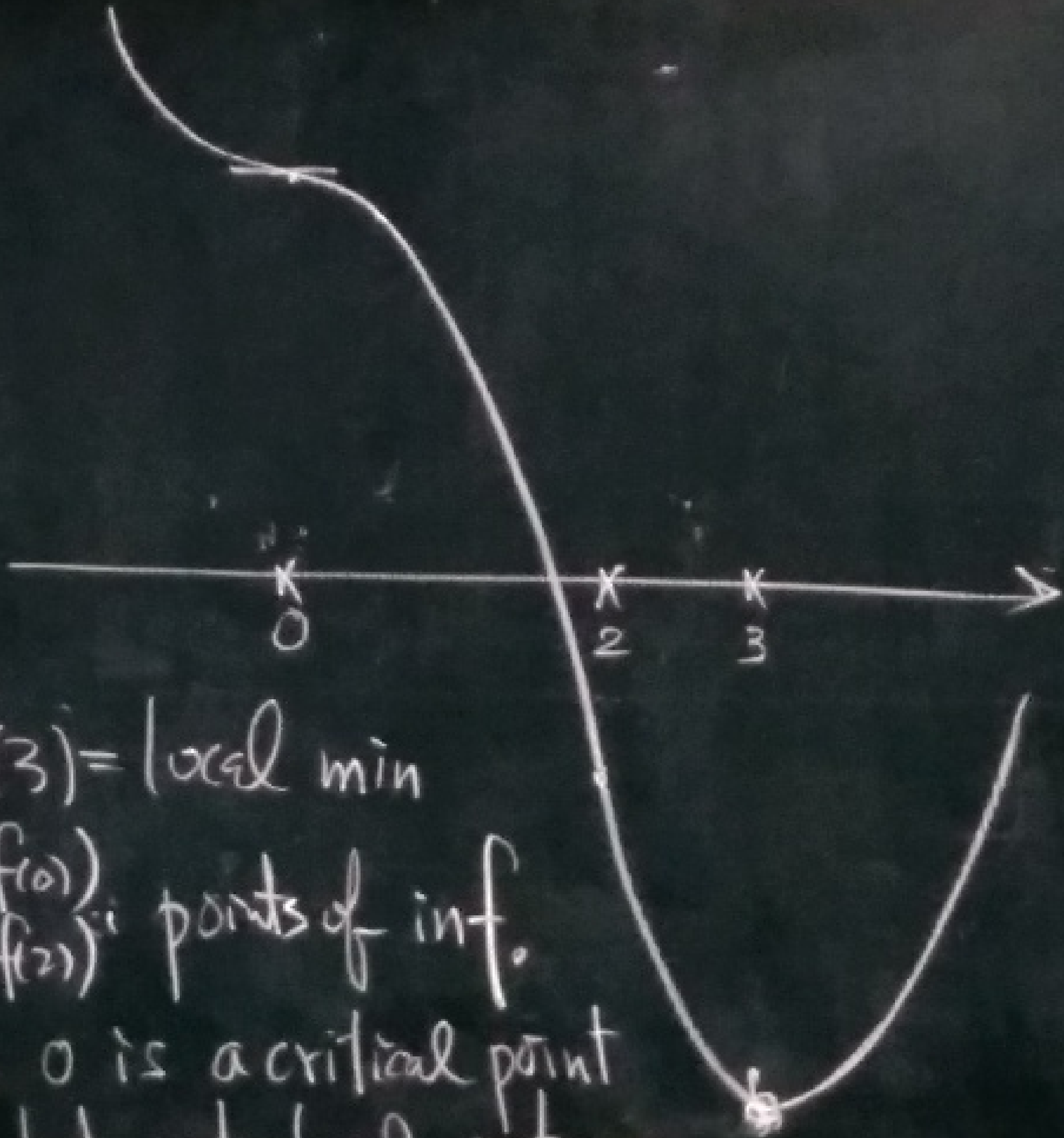
Sol. $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

$f(0) = 10$, $f(2) = -6$, $f(3) = -17$

Rewrite

x^2	+		+		+		+
$x-3$	−		−		−		+
$f'(x)$	−		−		−		+
		0		2		3 → local min	
$f''(x)$	+	infl	−	infl	+		+
x	−		+		+		+
$x-2$	−		−		+		+



$f(3)$ = local min

$(0, f(0))$, $(2, f(2))$: points of inf.

Rm 0 is a critical point but not local extreme.

Example: Sketch the graph of $f(x) = \dfrac{x^2+4}{2x}$

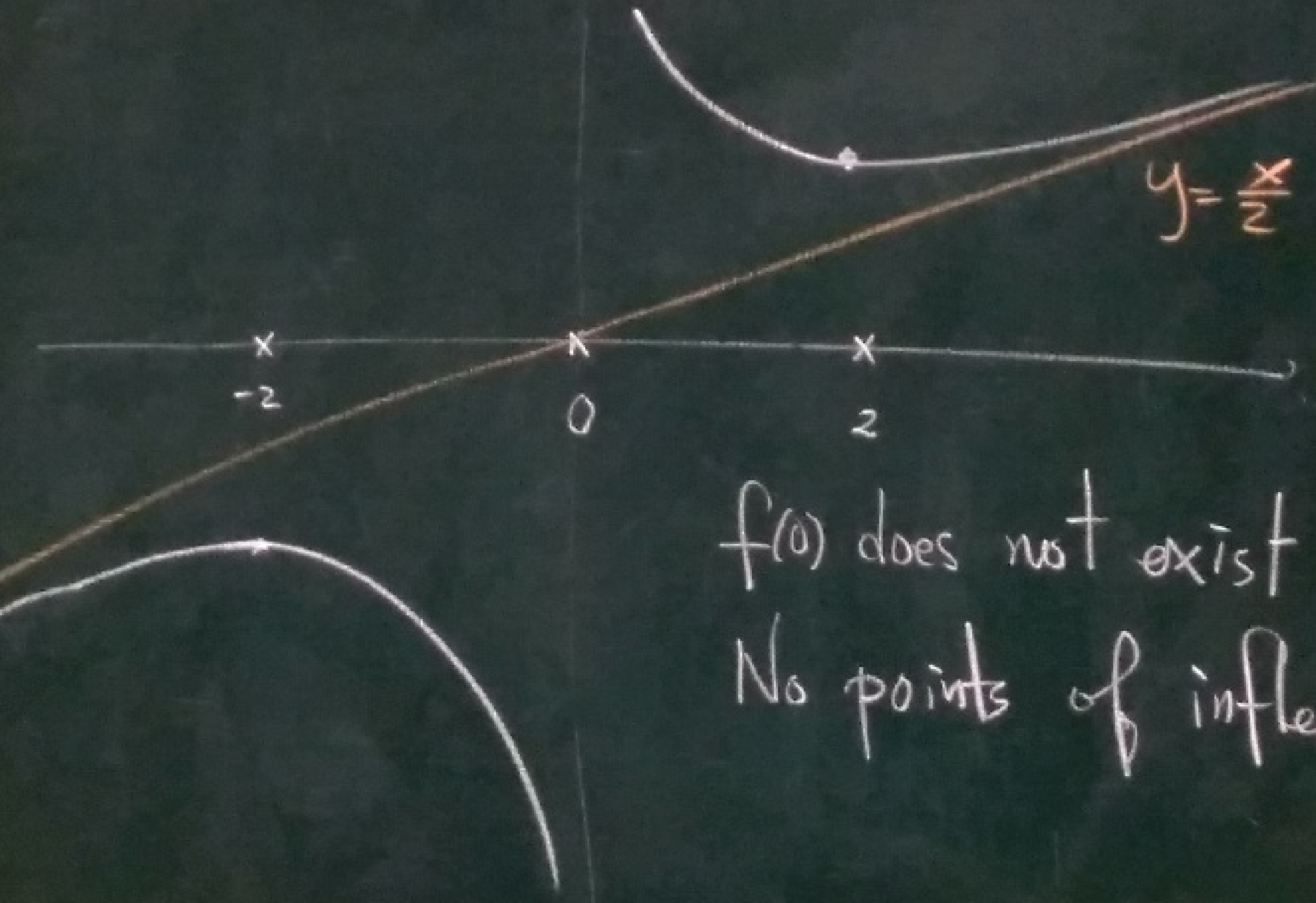
Sol $f(x) = \dfrac{x}{2} + \dfrac{2}{x}$

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{(x^2-4)}{2x^2}$$

$$f''(x) = 4x^{-3}$$

	$x<-2$	$-2<x<0$	$0<x<2$	$x>2$
x^2-4:	+	−	−	+
x^{-2}	+	+	+	+
f'	+	−	−	+
f'':	−	−	+	+
f	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

$f(-2) = -2$, $f(2) = 2$, $f(0)$ does not exist
= local max = local min



$f(0)$ does not exist
No points of inflection

L'Hôpital's Rule: a new technique to evaluate "$\lim \frac{0}{0}$" and "$\lim \frac{\infty}{\infty}$"

<u>Eg.</u> $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

<u>Thm</u> If f and g are cont. differentiable on $(a-\delta, a+\delta)$, $f(a)=g(a)=0$, $g'(x) \neq 0$ for $x \neq a$.

If $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$

then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$

Eg. $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = \lim\limits_{\theta \to 0} \dfrac{\cos\theta}{1} = 1$

Eg. $\lim\limits_{\theta \to 0} \dfrac{1-\cos\theta}{\theta^2} = \lim\limits_{\theta \to 0} \dfrac{(1-\cos\theta)'}{(\theta^2)'}$

$= \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{2\theta} = \dfrac{1}{2}$

Eg. $\lim\limits_{x \to 0} (1+ax)^{\frac{1}{x}}$ "1^∞"

$= \lim\limits_{x \to 0} e^{\left(\frac{\ln(1+ax)}{x}\right)}$ $\therefore$ Ans $:= e^a$

$\lim\limits_{x \to 0} \dfrac{\ln(1+ax)}{x} = \lim\limits_{x \to 0} \dfrac{\frac{a}{1+ax}}{1} = a$

Rm l'Hôpital Rule is applicable

to $\lim\limits_{\substack{x\to a \\ x\to a^{\pm} \\ x\to \pm\infty}}$ "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

Eg: $\lim\limits_{x\to\infty} \frac{x^{4.5}}{e^{\frac{x}{100}}}$

(Similarly for $\lim\limits_{x\to\infty} \frac{x^k}{e^{ax}}$, $k>0$, $a>0$)

$\frac{\infty}{\infty}$ $= \lim\limits_{x\to\infty} \frac{4.5x^{3.5}}{\frac{1}{100}e^{\frac{x}{100}}} = \cdots \lim\limits_{x\to\infty} \frac{** \; x^{-0.5}}{*** \; e^{\frac{x}{100}}}$

$= \frac{0}{\infty} = 0$