Rolle's Thm. Suppose $h(x)$ is cont. on $[a,b]$, diff. on (a,b) and $h(a)=h(b)$.

Then $\exists\, c\in(a,b)$ such that $h'(c)=0$.

pf. Suppose $h\begin{pmatrix} x_m \\ x_M \end{pmatrix} = \text{abs} \begin{matrix}\min \\ \max\end{matrix}$ on $[a,b]$

x_m, x_M can be (i) $c\in(a,b)$, $h'(c)=0$

~~(ii)~~ $c\in(a,b)$, $h'(c)$ not exist

(iii) $c = a$ or b

If either x_m or x_M is in (i), then c is found.

If both x_m, x_M are a or b, then $\min = h(x_m) = h(x_M) = \text{Max}$

$\Rightarrow h(x) = \text{constant}$, $h'(x)=0$, $\forall x\in(a,b)$. take any $c\in(a,b)$

The Mean Value Thm.

Suppose $f(x)$ is cont on $[a,b]$ diff on (a,b)

then $\exists c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Pf: Let $h(x) = f(x) - \left(f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)\right)$

$\Rightarrow h(x)$ is cont. on $[a,b]$. diff on (a,b).

$h(a) = 0$. $h(b) = 0$

$\xrightarrow{\text{Rolle}} 0 = h'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a}$ $\square$

Corollary 1. If $f'(x) \equiv 0$ on $(a.b)$
Then f is a constant on $(a.b)$.

Pf: Suppose not true

$\Rightarrow \exists\, x_1, x_2 \in (a,b).\ f(x_1) \neq f(x_2)$

$\xRightarrow{M.V.T.} \exists\, c \in (a,b).\ f'(c) = \dfrac{f(x_2)-f(x_1)}{x_2-x_1} \neq 0.$ Contradiction.

Corollary 2: If $f'(x) \equiv g'(x)$ on $(a.b)$
then $f(x) = g(x) + C$ on (a,b) for some constant C.

<u>Remark</u>. (Error of linear approximation)

If f'' exists on $(a-\delta, a+\delta)$.

$x \in (a-\delta, a+\delta)$. then $\exists\, c \in (a, x)$ or (x, a)

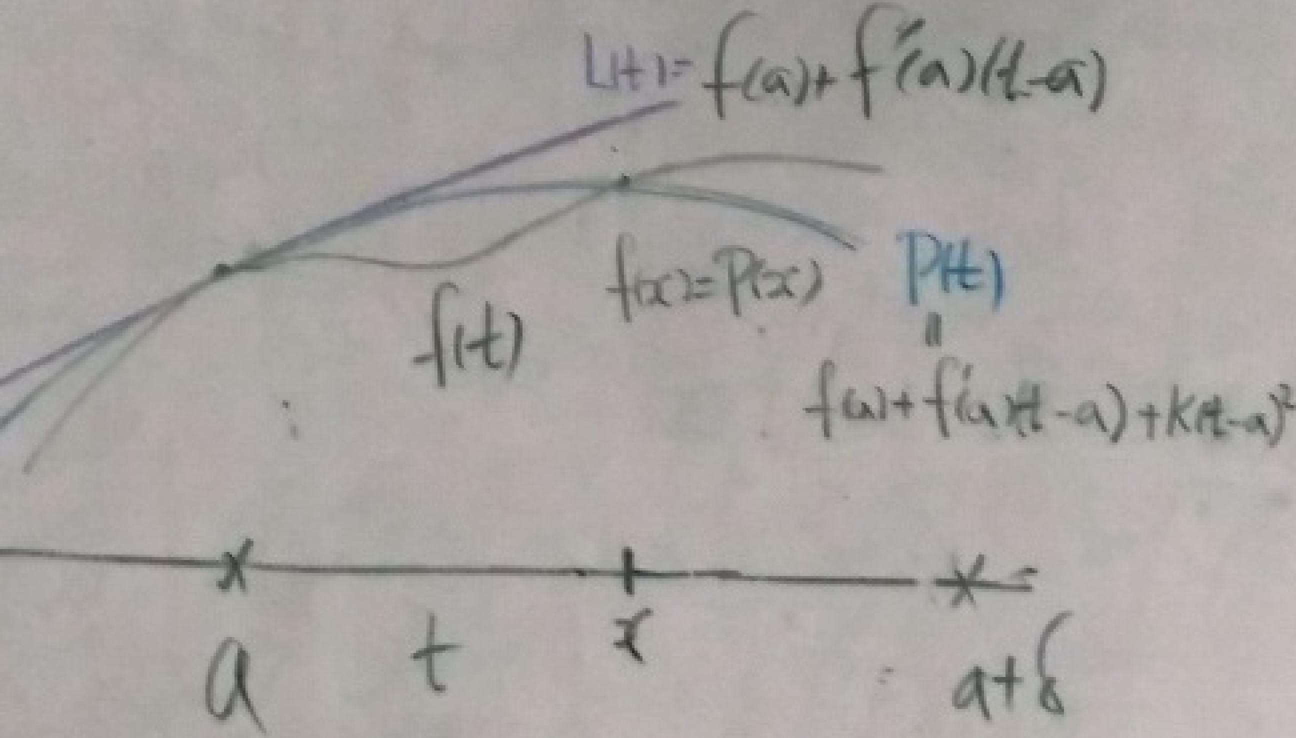
such that $f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$

<u>pf</u>: Suppose $a < x < a+\delta$, let $P : [a, x] \longrightarrow \mathbb{R}$

$$P(t) = \underline{f(a) + f'(a)(t-a)} + K(t-a)^2$$

Find K such that $P(x) = f(x)$

That is, $K = \dfrac{f(x) - (f(a) + f'(a)(x-a))}{(x-a)^2}$


$L(t) = f(a) + f'(a)(t-a)$
$f(x) = P(x)$
$P(t)$
$f(t)$
$= f(a) + f'(a)(t-a) + k(t-a)^2$
a
t
x
$a+b$

Note that

$$P(a) = f(a),\ P(x) = f(x)$$

$$P'(a) = f'(a)$$

Let $g(t) = f(t) - P(t)$

$$\Longrightarrow g(a) = 0,\ g(x) = 0,\ g'(a) = 0$$

$$\overset{\text{Rolle}}{\Longrightarrow} \exists\, c_1 \in (a, x),\ g'(c_1) = 0$$

$$g'(a) = g'(c_1) = 0 \overset{\text{Rolle}}{\Longrightarrow} \exists\, c \in (a, c_1),\ g''(c) = 0$$

$$\therefore\ 0 = g''(c) = f''(c) - P''(c) = f''(c) - 2k,\quad k = \frac{f''(c)}{2}$$

$$\therefore\ f(x) = P(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$$

$\square$

Cor 3 Suppose f is cont. on $[a,b]$, diff. on (a,b).

If $f'(x) \underset{<}{>} 0$ $\forall x \in (a,b)$ then f is $\underset{\text{decreasing}}{\text{increasing}}$ on $[a,b]$

($\underset{\text{decreasing}}{\text{increasing}}$: if $x_1 < x_2$ then $f(x_1) \underset{>}{<} f(x_2)$)

pf (" $>$ " and "increasing")

If not true (not increasing)

$\Rightarrow \exists\, x_1, x_2,\ a \le x_1 < x_2 \le b,\quad f(x_1) \ge f(x_2)$

$\overset{MVT}{\Longrightarrow} \exists\, c \in (x_1, x_2),\ f'(c) = \dfrac{f(x_2)-f(x_1)}{x_2 - x_1} \le 0$

contradiction $\square$

First derivative test for local extrema

Suppose f is cont. on $[a,b]$, c is a critical point and f is diff on $(a,b)\setminus\{c\}$

		left of c	right of c	
(i)	$f'(x)$	"+"	"$-$"	$\Rightarrow$ local max
(ii)	$f'(x)$	"$-$"	"+"	local min
(i)	$f(x)$	$\nearrow$	$\searrow$	
(ii)	$f(x)$	$\searrow$	$\nearrow$	
(iii)	$f'(x)$	"+"	"+"	not a local extrema
	$f'(x)$	"$-$"	"$-$"	not a local extrema
	$f(x)$	$\nearrow$	$\nearrow$	
	$f(x)$	$\searrow$	$\searrow$	

Eg2. Find all critical points of

$$f(x) = x^{\frac{1}{3}}(x-4) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$$

and determine whether they are local min/max or neither.

<u>Sol</u> $f'(x) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{\frac{-2}{3}} = \frac{4}{3}x^{\frac{-2}{3}}(x-1)$

critical points: 0, 1. ($f' = 0$ or does not exist)

$x^{\frac{-2}{3}}$	$+$		$+$		$+$
$(x-1)$	$-$		$-$		$+$
$f'(x)$	$-$		$-$		$+$
		0		1	
	$\searrow$		$\searrow$		$\nearrow$
		neither		local min.	