Linearization and differentials

(SKIP)

Def. If $f(x)$ is differentiable at $x=a$, then the linear function

$$L(x) = f(a) + f'(a)(x-a)$$

is called the linearization of (linear approximation) of $f(x)$ centered at a (i.e. near a)

(a is called the center of approximation)

Losely speaking, $L(x)$ is a linear function satisfying $f(x) \approx L(x)$ near a.

Example: find approximate value of $\sqrt{1.001}$.

Sol: consider $f(x) = \sqrt{1+x}$ near $x=0$, then

$$L(x) = f(0) + f'(0)(x-0)$$

and $\sqrt{1.001} \cong L(0.001)$

$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$$

$$f'(0) = \frac{1}{2}$$

$$\therefore \sqrt{1.001} \cong 1 + \frac{1}{2}0.001$$

$$= 1.0005$$

In fact $(1+x)^k \cong 1+kx$

near $x=0$

Example. find approximate values of

(i) $(7.97)^{\frac{1}{3}}$. (ii) $\sin\left(\frac{\pi}{6}+0.001\right)$

Sol $(7.97)^{\frac{1}{3}} = (8-0.03)^{\frac{1}{3}}$

$= \left(8\left(1-\frac{0.03}{8}\right)\right)^{\frac{1}{3}}$

$= 2\left(1-\frac{0.03}{8}\right)^{\frac{1}{3}}$

$\cong 2\left(1+\frac{1}{3}\left(\frac{-0.03}{8}\right)\right)$

$= 2\left(1-\frac{0.01}{8}\right)$

$= 2-0.0025 = 1.9975$

(ii) $f(x) = \sin x$

$a = \frac{\pi}{6}$

$x - a = 0.001$

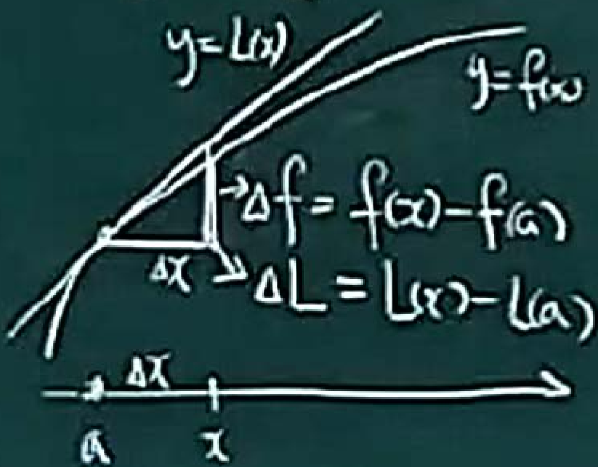
$\therefore f(x) \cong L(x)$

$= L(a) + f'(a)(x-a)$

Note: $L(a) = f(a)$

$= \sin(\frac{\pi}{6}) + \cos(\frac{\pi}{6}) \cdot 0.001$

$= \frac{1}{2} + \frac{\sqrt{3}}{2} 0.001$



$f(x) \cong L(x)$ near a

$f(x) - f(a) \cong L(x) - L(a)$

$(L(a) = f(a))$

$\Delta f \cong \Delta L = L(x) - f(a) = f'(a)\underset{\underset{x-a}{\parallel}}{\Delta x}$

$\Delta f = \Delta L + \text{error}$

$f(x) - f(a) = f'(a)(x-a) + \text{error}$ (*)

<u>Prop</u> If f is diff. at $x=a$

then (not only $\lim\limits_{\Delta x \to 0} (\text{error}) = 0$)

$$\lim_{\Delta x \to 0} \left(\frac{\text{error}}{\Delta x} \right) = 0$$

where $\text{error} = f(x) - f(a) - f'(a)(x-a)$

$= f(x) - L(x)$

pf:

$$\lim_{\Delta x \to 0} \frac{\text{error}}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{x-a}$$

$\Delta x = x - a$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a} - f'(a) = 0$$

end of proof

$(*) \Rightarrow$

$$\therefore f(x) - f(a) = f'(a)(x-a) + \frac{\text{error}}{(x-a)}(x-a)$$

$$\left[= f'(a)(x-a) + \begin{pmatrix} \text{something that} \\ \text{goes to zero} \\ \text{when } x \to a \ (\Delta x \to 0) \end{pmatrix} (x-a) \right]$$

or

$$\left[\begin{array}{l} = f'(a)(x-a) + \varepsilon \cdot (x-a) \\ \text{where } \lim_{x \to a} \varepsilon = 0 \end{array} \right]$$

In sammary:

$f(x)$ is differentiable at a

$$\Longleftrightarrow \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(a) \quad \begin{pmatrix} \Delta f = f(x) - f(a) \\ \Delta x = x - a \end{pmatrix}$$

$$\Longleftrightarrow \frac{\Delta f}{\Delta x} = f'(a) + \varepsilon, \quad \lim_{\Delta x \to 0} \varepsilon = 0$$

$$(\Delta f = f(x) - f(a). \quad \Delta x = x - a)$$

$$\Longleftrightarrow \Delta f = f'(a)\Delta x + \varepsilon \Delta x, \quad \lim_{\Delta x \to 0} \varepsilon = 0$$

Proof of Chain rule:

$u = g(x) \qquad u_0 = g(x_0)$

$y = f(g(x))$

$\Delta x = x - x_0$

$\Delta u = g(x) - g(x_0) = u - u_0$

$\Delta y = f(g(x)) - f(g(x_0)) = f(u) - f(u_0)$

Old proof:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$\Big\downarrow \Delta x \to 0$$

$$\frac{d}{dx} f(g(x_0)) = f'(g(x_0)) \cdot g'(x_0)$$

problem: if $\Delta u = 0$?

$$\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = f'(u_0)$$

$$\Leftrightarrow \Delta y = f'(u_0)\Delta u + \varepsilon_2 \Delta u$$

where $\lim_{\Delta u \to 0} \varepsilon_2 = 0$

$$\Delta u = g'(x_0)\Delta x + \varepsilon_1 \Delta x$$

$$\lim_{\Delta x \to 0} \varepsilon_1 = 0 \quad \left(\lim_{\Delta x \to 0} \varepsilon_2 = \lim_{\Delta u \to 0} \varepsilon_2 = 0 \right)$$

$\therefore \Delta y = (f'(u_0) + \varepsilon_2)\Delta u$

$= (f'(u_0) + \varepsilon_2)(g'(x_0) + \varepsilon_1)\Delta x$

$\lim_{\Delta x \to 0} \varepsilon_1, \varepsilon_2 = 0$

$= f'(u_0) g'(x_0) \Delta x$

$+ (\underbrace{f'(u_0)\varepsilon_1 + g'(x_0)\varepsilon_2 + \varepsilon_1\varepsilon_2}_{\parallel})\Delta x$

$= \left(\frac{dy}{dx}\Big|_{x=x_0}\right)\Delta x + \varepsilon \Delta x$

where $\lim_{\Delta x \to 0} \varepsilon = 0$

$\therefore \frac{dy}{dx}\Big|_{x=x_0} = f'(u_0) g'(x_0)$

i.e. $\frac{d}{dx} f(g(x_0)) = f'(g(x_0)) g'(x_0)$