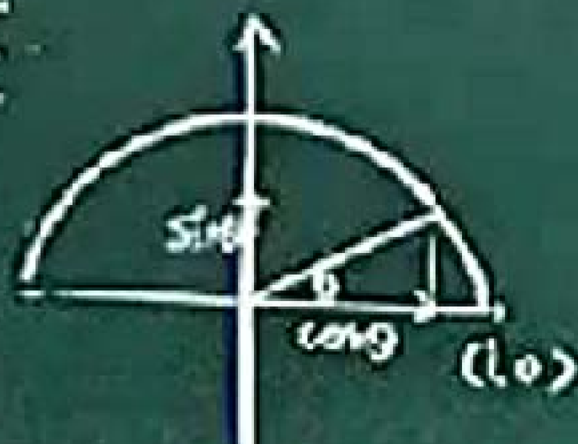
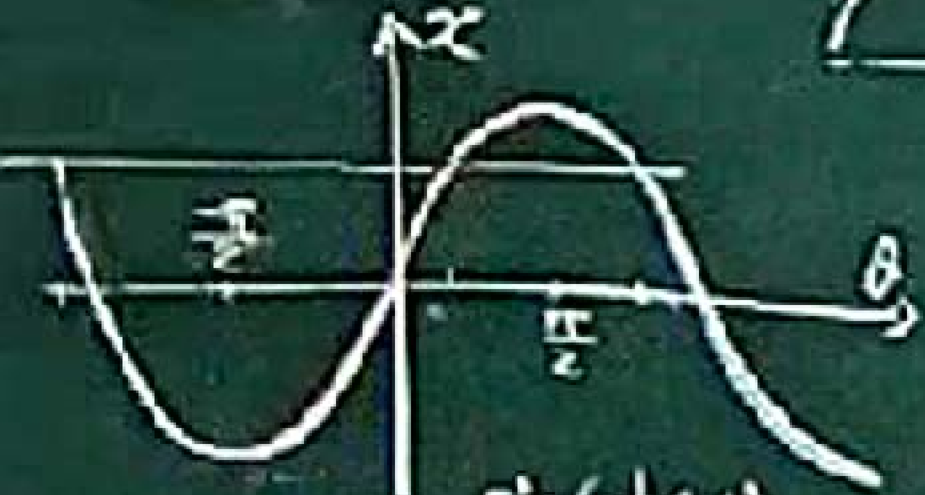
Inverse trigonometric functions

* Inverse functions of sin, cos, tan, cot, sec, csc.

<u>definition of $\sin^{-1}x$</u>

$x=\sin\theta$.





$$\theta\in\mathbb{R} \xrightarrow{\sin\ (\text{not } 1\text{-}1)} x\in[-1,1]$$

$$\xleftarrow[\sin^{-1}?]{}$$

Restrict ↓

$$\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \xrightarrow{\sin\ (1\text{-}1)} x\in[-1,1]$$

$$\xleftarrow[\sin^{-1}]{}$$

$\sin(\sin^{-1}x) = x, \ \forall x\in[-1,1]$

$\sin^{-1}(\sin\theta) = \theta, \ \forall \theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$\theta=\sin^{-1}x$ and $x=\sin\theta$
are inverse function to each other

$$\begin{tikzcd}
\theta\in[-\frac{\pi}{2},\frac{\pi}{2}] \arrow[r, shift left, "\sin"] \arrow[d, phantom, "\parallel"] & x\in[-1,1] \arrow[l, shift left, "\sin^{-1}"] \arrow[d, phantom, "\parallel"] \\
D_{\sin} \arrow[d, phantom, "\parallel"] & R_{\sin} \arrow[d, phantom, "\parallel"] \\
R_{\sin^{-1}} & D_{\sin^{-1}}
\end{tikzcd}$$

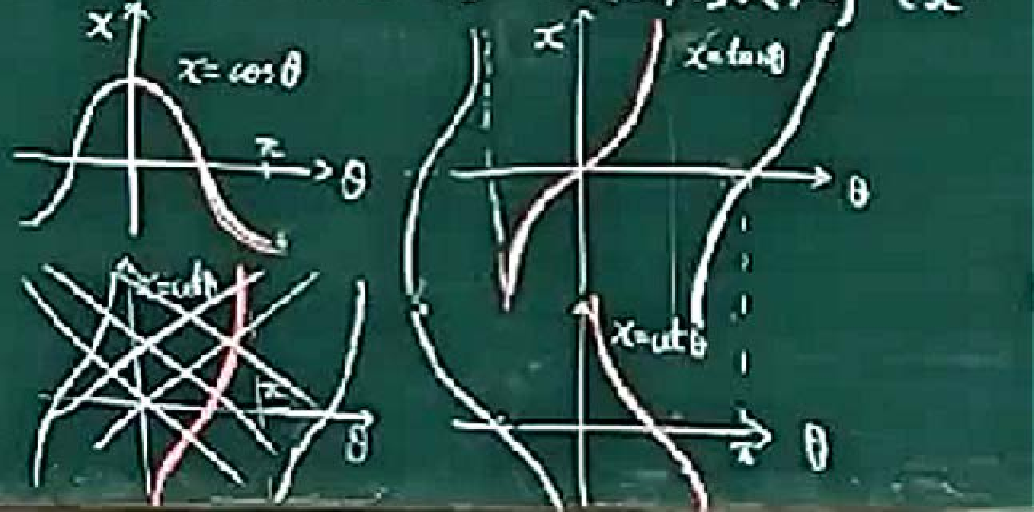
That is, if $x\in \overset{[-1,1]}{\cancel{\mathbb{R}}}$, then
$\sin^{-1}x$ is the unique $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$
Satisfying $\sin\theta=x$

<u>Note</u>: $\sin^{-1}(\sin\pi)\neq\pi$

$\sin\pi=0$

$\theta=\sin^{-1}0 \Leftrightarrow \begin{cases}\sin\theta=0\\ \theta\in[-\frac{\pi}{2},\frac{\pi}{2}]\end{cases} \therefore \theta=0\neq\pi$

$f(\cdot)$	$D_f = R_{f^{-1}} \underset{f^{-1}}{\overset{f}{\rightleftarrows}}$	$R_f = D_{f^{-1}}$	$f^{-1}(\ )$
sin	$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	$x \in [-1, 1]$	$\sin^{-1}$
cos	$\theta \in [0, \pi]$	$x \in [-1, 1]$	$\cos^{-1}$
tan	$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$x \in \mathbb{R}$	$\tan^{-1}$
cot	$\theta \in (0, \pi)$	$x \in \mathbb{R}$	$\cot^{-1}$
sec	$\theta \in [0, \pi] \setminus \{\frac{\pi}{2}\}$	$x \in (-\infty, -1] \cup [1, \infty)$	$\sec^{-1}$
csc	$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$	(i.e. $\lvert x \rvert >= 1$) $x \in (-\infty, -1] \cup [1, \infty)$	$\csc^{-1}$



Derivatives of inverse trigonometric functions:

(1) $\frac{d}{dx}\sin^{-1}x = \dfrac{1}{\frac{d\sin\theta}{d\theta}} = \frac{1}{\cos\theta}$

where $x = \sin\theta$

Since $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\sqrt{1-x^2}$

but $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\Rightarrow \cos\theta \geq 0$ $\therefore \pm$, take "+"

$\therefore \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ $x \in (-1, 1)$

(2) Similarly, $\frac{d}{dx}\cos^{-1}x = \frac{1}{-\sin\theta}$, $\theta \in [0, \pi]$

$(\cos\theta = x)$

$= \frac{-1}{\sqrt{1-x^2}}$ $x \in (-1, 1)$

(3) $\frac{d}{dx}\tan^{-1}x = \frac{1}{\frac{d}{d\theta}\tan\theta} = \frac{1}{\sec^2\theta}$

$(\tan\theta = x)$ $= \frac{1}{1+x^2}$, $x \in \mathbb{R}$

(4) $\frac{d}{dx}\cot^{-1}x$: exercise.

(6) $\frac{d}{dx}\csc^{-1}x = \frac{1}{\frac{d}{d\theta}\csc\theta}$

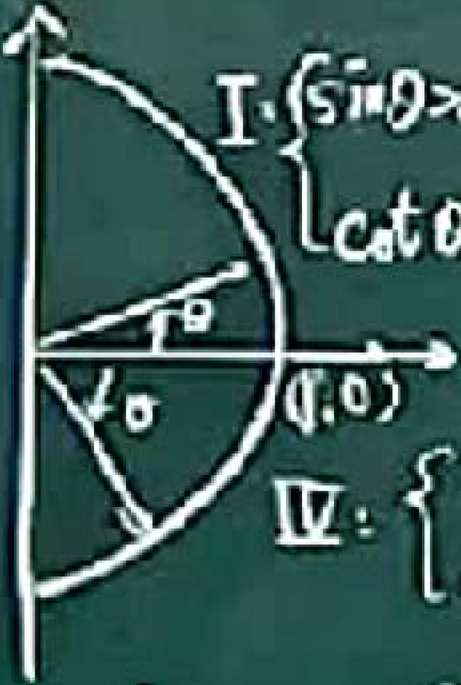
$(\csc\theta = x)$

$= \frac{1}{\frac{d}{d\theta}(\sin\theta)^{-1}} = \frac{1}{-(\sin\theta)^{-2}\cos\theta}$

$= \frac{1}{-\frac{\cos\theta}{\sin\theta}\frac{1}{\sin\theta}} = \frac{-1}{\csc\theta\cot\theta}$

$\cot^2\theta = \csc^2\theta - 1 = x^2 - 1$

$\cot\theta = \pm\sqrt{x^2-1}$



$$\mathrm{I}: \begin{cases} \sin\theta > 0, & x = \csc\theta > 0 \\ \cot\theta > 0, & \text{on } 0 < \theta < \frac{\pi}{2} \end{cases}$$

$$\mathrm{IV}: \begin{cases} \sin\theta < 0, & x = \csc\theta < 0 \\ \cot < 0 & \text{on } -\frac{\pi}{2} < \theta < 0 \end{cases}$$

$\cot\theta > 0$, if $x > 0$

$\cot\theta < 0$, if $x < 0$.

ie $\cot\theta = \underbrace{\left(\frac{x}{|x|}\right)}_{\pm}\sqrt{x^2-1}$

$$\therefore \frac{d}{dx}\csc^{-1}x = \frac{-1}{x\,\frac{x}{|x|}\sqrt{x^2-1}}$$

$$= \frac{-1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$