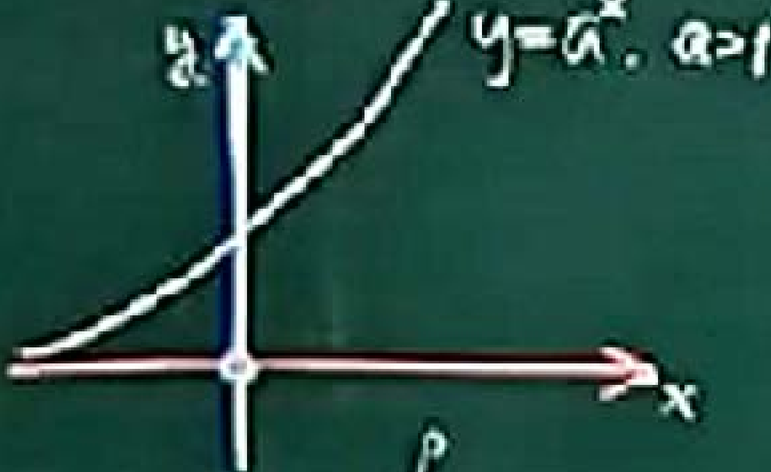
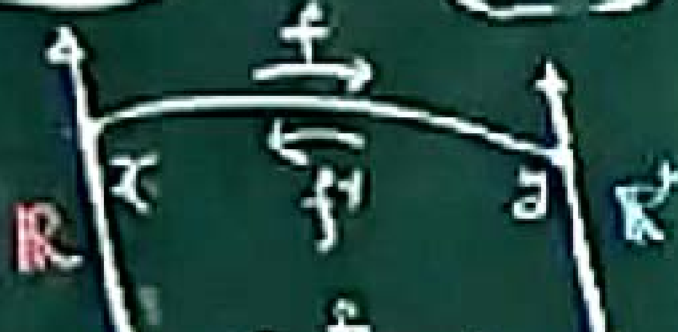
Derivative of inverse functions and Logarithm



$$x \in \mathbb{R} \xmapsto{f} y = a^x > 0$$



$$x \xrightarrow{f} y$$

$$x \xleftarrow{f^{-1}} y$$

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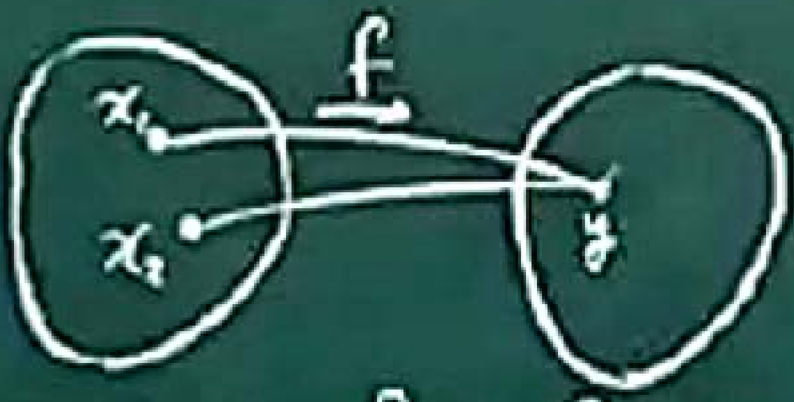
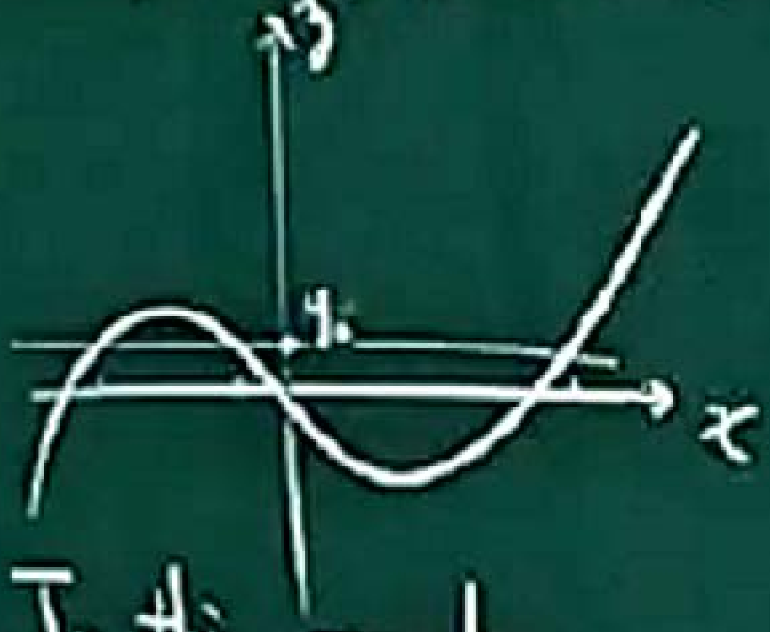


If $x_1 \neq x_2$, $f(x_1)=f(x_2)=y$

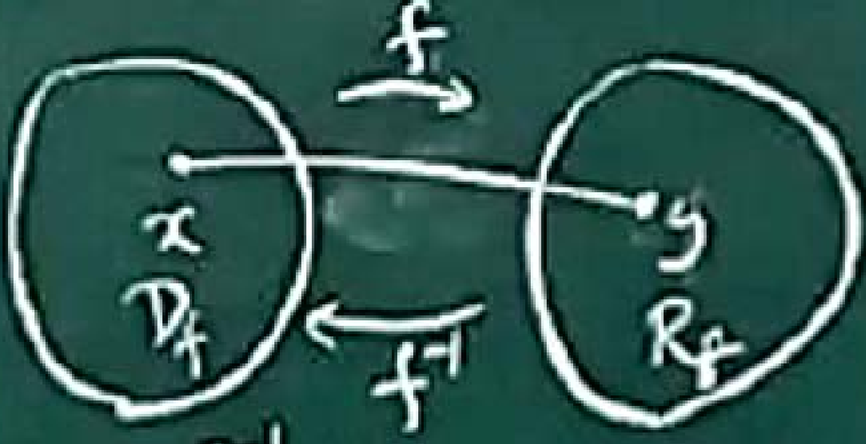
, then we can not define $f^{-1}(y)$!

<u>Example</u>. $f(x)=x^3-3x^2-1$



In this example $f^{-1}(y_1)$ is not defined

$f: D_f \longrightarrow R_f$



Then $f^{-1}: R_f \longrightarrow D_f$ exists

$\Longleftrightarrow \begin{cases} f \text{ is \underline{one to one} on } D_f \\ f \text{ maps } D_f \text{ \underline{onto} } R_f \end{cases}$

One to one: $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$

onto: $R_f = \{f(x), x \in D_f\}$

Proposition: $f: D_f \longmapsto R_f$.

if f^{-1} exists, then

(i) $f^{-1}(f(x)) = x$ for any $x \in D_f$

(ii) $f(f^{-1}(y)) = y$ for any $y \in R_f$

<u>Example</u>: The inverse function of

$y = a^x$ is $x = \log_a y$

$(y = \log_a x)$

ie

$$\log_a(a^x) = x, \ \forall x \in \mathbb{R}$$

$$a^{\log_a y} = y, \ \forall y \in \mathbb{R}^+$$

In particular, when $a =$ Euler number e

$$\log_e(\cdot) \stackrel{\text{def}}{=} \ln(\cdot)$$

(natural log)

$$\ln e^x = x, \ \forall x \in \mathbb{R}$$

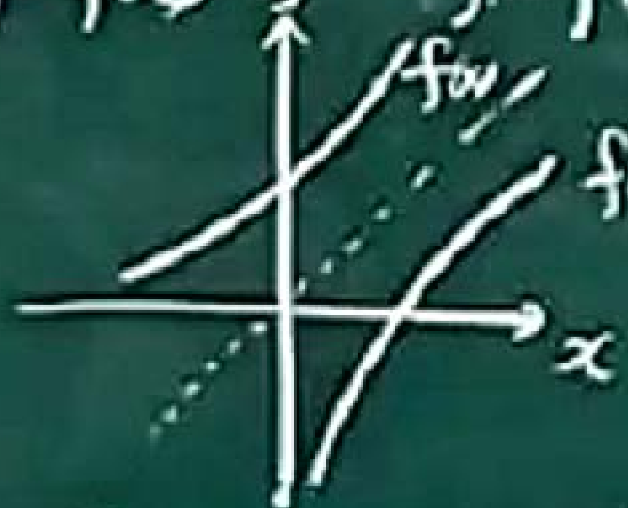
$$e^{\ln y} = y, \ \forall y \in \mathbb{R}^+$$

Notation difference

original function   inverse function

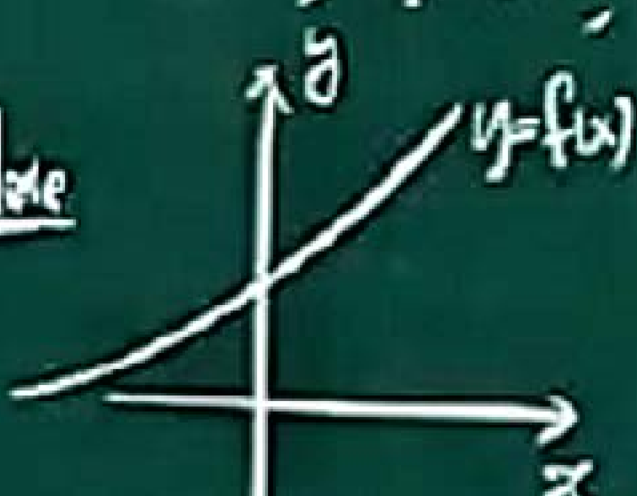
$y=f(x)$ $y=f^{-1}(x)$

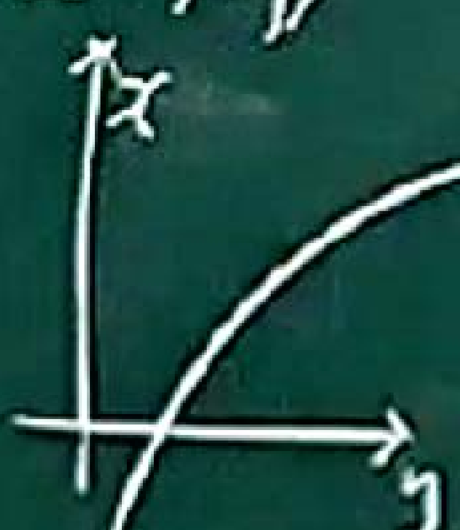
<u>textbook</u>



$y=f(x)$ $x=f^{-1}(y)$

<u>Here</u>





$f^{-1}(f(x)) = x \quad \forall x \in D_f$

$\frac{d}{dx} \Rightarrow \left.\frac{df^{-1}(y)}{dy}\right|_{y=f(x)} \cdot \frac{df(x)}{dx} = 1$

$$\left.\frac{df^{-1}(y)}{dy}\right|_{y=f(x)} = \frac{1}{\frac{df(x)}{dx}}$$

or $$\frac{df^{-1}(y)}{dy} = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=f^{-1}(y)}}$$

<u>Textbook:</u>

$$\left.\frac{d f^{-1}(x)}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=f^{-1}(b)}}$$

<u>Note</u>: the inverse function of $e^{(\cdot)}$ is $\ln(\cdot)$

$$\boxed{\begin{array}{c} \dfrac{d}{dx}\ln x = \dfrac{1}{\frac{d}{dx}e^{x}} = \dfrac{1}{e^{x}} \\ \text{(WRONG)} \end{array}}$$

Correct formula: $y = e^{x},\ x \in \mathbb{R}$

$x = \ln y,\ y \in \mathbb{R}^{+}$

$$\frac{d}{dy}\ln y = \frac{1}{\frac{d}{dx}e^{x}} = \frac{1}{e^{x}} = \frac{1}{y}.$$

$$\frac{y = e^{x}}{x = \ln y} \qquad (y>0)$$

$$\left(\frac{d}{dy}\ln y \Big|_{y=e^{x}} = \frac{1}{e^{x}} \right)$$

Conclusion:

$$\frac{d}{dy}\ln y = \frac{1}{y}, \quad y>0$$

i.e. $\frac{d}{dx}\ln x = \frac{1}{x}, \quad x>0$

Application: if $u(x)>0$

then $\frac{d}{dx}\ln u(x) = \frac{1}{u(x)}\cdot u'(x)$

Example: $\frac{d}{dx}\ln(x^2+3) = \frac{2x}{x^2+3}$

Remark: If $x<0$

$$\frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$$

$\therefore \frac{d}{dx}\ln|x| = \frac{1}{x}, \quad x\neq 0$

Example: $a>0$, $\frac{d}{dx}a^x = ?$

Sol: $a = e^{\ln a}$

$$\frac{d}{dx}a^x = \frac{d}{dx}\left(e^{\ln a}\right)^x$$

$$= \frac{d}{dx}\left(e^{x\ln a}\right)$$

$$= \left(e^{x\ln a}\right)\frac{d}{dx}(x\ln a)$$

$$= a^x \cdot \ln a$$

Note: we showed that

$$\frac{d}{dx}a^x = g(a)\,a^x$$

where $g(a) = \underbrace{\lim_{h\to 0}\frac{a^h-1}{h}}_{=\,\ln(a)}$

Now we know

Application: $a>0$, $a\neq 1$

$$\frac{d}{dx}a^{u(x)} = \underline{\ln a \cdot a^{u(x)}} \cdot u'(x)$$

Example: $\frac{d}{dx}3^{\sin x}$

$$= (\ln 3)(3^{\sin(x)})\cos x.$$

Example. $a>0$, $a\neq 1$, $x>0$

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\left(\frac{\log_e x}{\log_e a}\right)$$

$$= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{x\ln a}$$

Application: $a>0$, $u(x)>0$

$$\frac{d}{dx}\log_a u(x) = \frac{u'(x)}{u(x)\ln a}.$$

Example: $y = \dfrac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}$, $x > 1$

$\dfrac{dy}{dx} = ?$

Sol. $\ln y = \ln\left(\dfrac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}\right)$

$= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$

$\xRightarrow{\frac{d}{dx}} \dfrac{y'}{y} = \dfrac{2x}{x^2+1} + \dfrac{1}{2}\dfrac{1}{(x+3)} - \dfrac{1}{x-1}$

$\therefore y' = y \cdot$ (right hand side)

$= \dfrac{(x^2+1)(x+3)^{\frac{1}{2}}}{(x-1)} \cdot \left(\dfrac{2x}{x^2+1} + \dfrac{1}{2}\dfrac{1}{x+3} - \dfrac{1}{x-1}\right)$

Application: $\underline{x>0}$, $n \in \mathbb{R}$

$$\frac{d}{dx} x^n = ? \quad (x = e^{\ln x})$$

$$= \frac{d}{dx}\left(e^{\ln x}\right)^n = \frac{d}{dx}\left(e^{n \ln x}\right)$$

$$= \left(e^{n \ln x}\right) \cdot \frac{n}{x}$$

$$= x^n \cdot \frac{n}{x} = n x^{n-1}$$

(Ignore the $x<0$ part in textbook)

Example: $x>0$, $\frac{d}{dx} x^x = ?$

$\underline{\text{Ans.}}$ $= \frac{d}{dx}\left(e^{\ln x}\right)^x = \frac{d}{dx} e^{x \ln x}$

$$= e^{x \ln x} \cdot \left(\ln x + x \cdot \frac{1}{x}\right)$$

$$= x^x (\ln x + 1)$$

Application: $e = \lim\limits_{x \to 0} (1+x)^{\frac{1}{x}}$

证: let $f(x) = \ln x$, $x > 0$

$$f'(1) = \frac{1}{x}\Big|_{x=1} = 1.$$

$$\parallel$$

$$\lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h}$$

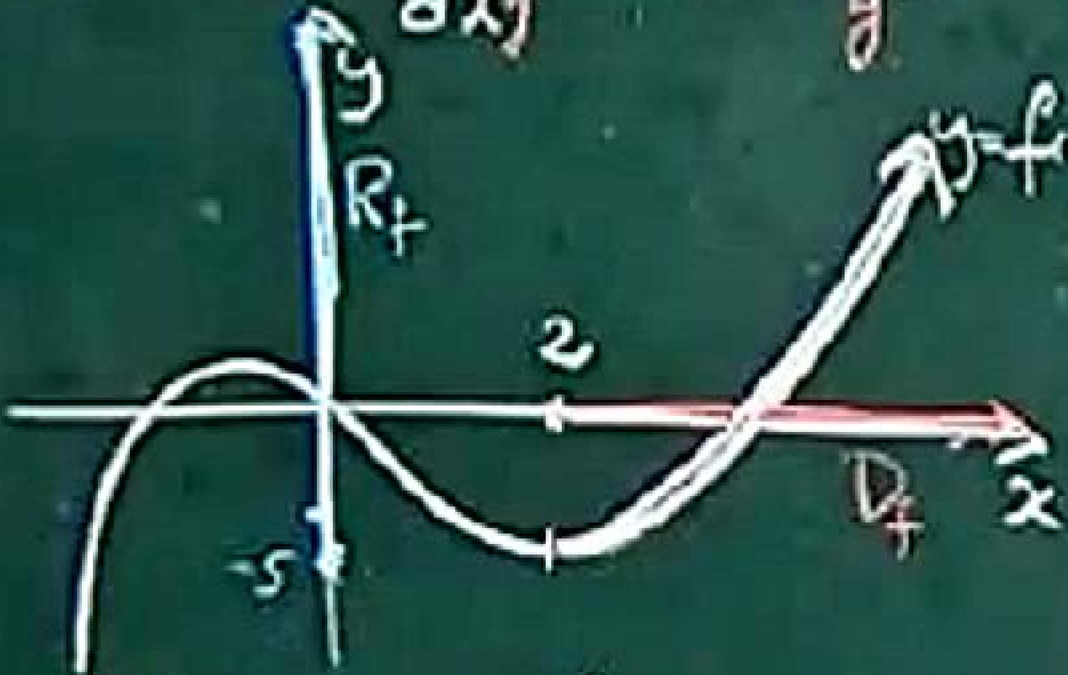
$$= \lim_{h \to 0} \frac{\ln(1+h)}{h}$$

$$= \lim_{h \to 0} \ln(1+h)^{\frac{1}{h}}$$

$$= \lim_{x \to 0} \ln(1+x)^{\frac{1}{x}}$$

Example: $f(x) = x^3 - 3x^2 - 1$, $x \ge 2$.

Find $\dfrac{df^{-1}(y)}{dy}$ at $y = -1 = f(3)$



Ans: $\left.\dfrac{df^{-1}(y)}{dy}\right|_{y=-1} = \dfrac{1}{\left.\dfrac{df(x)}{dx}\right|_{x=3}}$

$= \left.\dfrac{1}{3x^2-6x}\right|_{x=3} = \dfrac{1}{9}$