Application of Chain Rule

* $\frac{d}{dx}(u(x))^n = n\,u^{n-1}(x) \cdot u'(x)$
* $\frac{d}{dx} e^{u(x)} = e^{u(x)} \cdot u'(x)$
* $\frac{d}{dx} \sin u(x) = \{\cos u(x)\}\, u'(x)$

etc.

<u>Example</u>: $\frac{d}{dx}\left(\frac{1}{v(x)}\right) = \frac{d}{dx}(v(x))^{-1}$

$\left(= \frac{d}{dx} g(v(x)),\ g(v) = v^{-1}\right)$

$= -v^{-2}(x) \cdot v'(x) = \frac{-v'(x)}{v^2(x)}$

<u>Example</u>: $\frac{d}{dx}\sqrt{1+x^2} = \frac{d}{dx}(1+x^2)^{\frac{1}{2}}$

$= \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{1+x^2}}$

Example: $\frac{d}{dx} e^{e^x}$

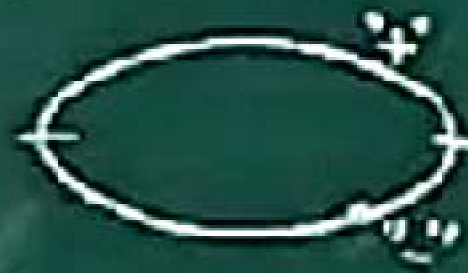
$= e^{e^x} \cdot e^x$

Implicit differentiation

Example: Find the tangent and normal lines of $\frac{x^2}{4}+y^2=1$ at $\left(\sqrt{2}, \frac{-1}{\sqrt{2}}\right)$

Sol: Method (a) (explicit diff.)

$y^2 = 1-\frac{x^2}{4}$



$\Rightarrow y = \pm\sqrt{1-\frac{x^2}{4}}$

take "−"

$\frac{d}{dx}\left(-\sqrt{1-\frac{x^2}{4}}\right) = \frac{d}{dx}\left(-\left(1-\frac{x^2}{4}\right)^{\frac{1}{2}}\right)$

$= \frac{-1}{2}\left(1-\frac{x^2}{4}\right)^{\frac{-1}{2}} \cdot \left(\frac{-x}{2}\right)$

$\therefore$ Slope of tangent

$$= \frac{-1}{2}\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}\left(\frac{-x}{2}\right)\Bigg|_{x=\sqrt{2}}$$

$$= \frac{\sqrt{2}}{4}\left(\frac{1}{2}\right)^{-\frac{1}{2}} = \frac{1}{2}$$

tangent: $\left(\dfrac{y-\frac{-1}{\sqrt{2}}}{x-\sqrt{2}}\right) = \dfrac{1}{2}$

normal: $\left(\dfrac{y-\frac{-1}{\sqrt{2}}}{x-\sqrt{2}}\right) = -2$

Method (b): (implicit diff.)

$\frac{x^2}{4}+y^2=1$ implicitly defines a function $y=y(x)$ near $\left(\sqrt{2}, \frac{-1}{\sqrt{2}}\right)$

i.e. $\frac{x^2}{4}+y^2(x)=1$

$$\frac{x^2}{4}+y^2(x)=1 \quad \text{for all } x \text{ near } \sqrt{2}$$

$$\xRightarrow{\frac{d}{dx}} \frac{x}{2}+2y(x)\cdot y'(x)=0$$

Substitute $x \leftarrow \sqrt{2}$, $y(x) \leftarrow \frac{1}{\sqrt{2}}$

$$\Rightarrow y'=\frac{-x}{4y(x)}=-\frac{1}{2}$$

$\Rightarrow$ tangent line, normal line.

<u>Example</u>: $\frac{(x+y)^2}{32}+\frac{(x-y)^2}{8}=1$ implicitly defines $y=y(x)$ near $(3,1)$. find $y'(3)$ and $y''(3)$

<u>Sol</u>: $y'(3)$.

$$\xRightarrow{\frac{d}{dx}} \frac{(x+y)(1+y')}{16}+\frac{(x-y)(1-y')}{4}=0$$

$$\xRightarrow{(x,y)=(3,1)} \frac{4(1+y')}{16}+\frac{2(1-y')}{4}=0, \; y'(3)=3$$

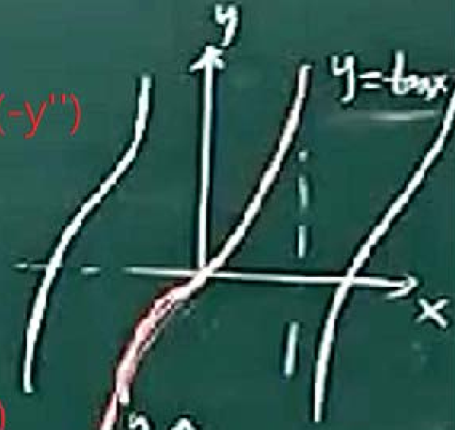
$$y''(3):\ \frac{d}{dx}\left(\frac{(x+y)(1+y')}{16}+\frac{(x-y)(1-y')}{4}=0\right)$$

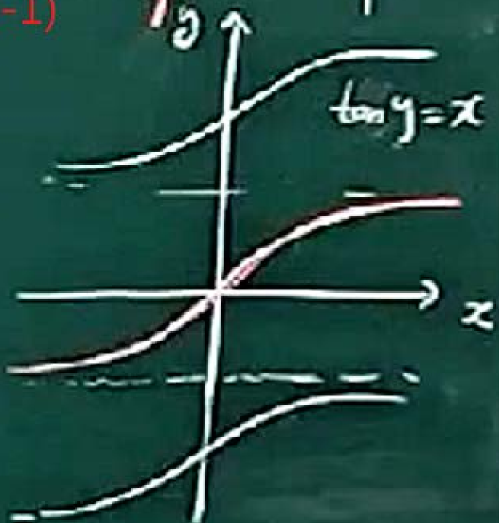
$$\Rightarrow \frac{(1+y')(1+y')+(x+y)y''}{16}+\frac{(1-y')(1-y')+(x-y)y''}{4}=0 \dashrightarrow (-y'')$$

substitute $x=3,\ y(3)=1,\ y'(3)=3$

$$\frac{(1+3)^2+(1+3)y''}{16}+\frac{(1-3)^2+(1-3)y''}{4}=0 \dashrightarrow -(3-1)$$

$$1+1=-\frac{y''}{2}-\frac{y''}{4},\quad y''(3)=8$$





Example: $x=\tan y,\ \frac{dy}{dx}=?$

$$\Rightarrow 1=(\sec^2 y)\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+x^2}$$