

Application of Chain Rule

$$* \frac{d}{dx}(u(x))^n = n u^{n-1}(x) \cdot u'(x)$$

$$* \frac{d}{dx} e^{u(x)} = e^{u(x)} \cdot u'(x)$$

$$* \frac{d}{dx} \sin u(x) = (\cos u(x)) u'(x)$$

etc.

Example: $\frac{d}{dx} \left(\frac{1}{\sqrt{u(x)}} \right) = \frac{d}{dx} (u(x))^{-1/2}$

$$\left(= \frac{d}{dx} g(u(x)), g(u) = u^{-1/2} \right)$$
$$= -u^{-3/2}(x) \cdot u'(x) = \frac{-u'(x)}{u^{3/2}(x)}$$

Example: $\frac{d}{dx} \sqrt{1+x^2} = \frac{d}{dx} (1+x^2)^{1/2}$

$$= \frac{1}{2} (1+x^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{1+x^2}}$$

Example: $\frac{d}{dx} e^{e^x}$
 $= e^{e^x} \cdot e^x$

Implicit differentiation

Example: Find the tangent and normal lines of $\frac{x^2}{4} + y^2 = 1$ at $(\sqrt{2}, \frac{1}{\sqrt{2}})$

Sol. Method (a) (explicit diff.)

$$y^2 = 1 - \frac{x^2}{4}$$

$$\Rightarrow y = \pm \sqrt{1 - \frac{x^2}{4}}$$

take "-"



$$\begin{aligned} \frac{d}{dx} \left(-\sqrt{1 - \frac{x^2}{4}} \right) &= \frac{d}{dx} \left(-\left(1 - \frac{x^2}{4} \right)^{\frac{1}{2}} \right) \\ &= -\frac{1}{2} \left(1 - \frac{x^2}{4} \right)^{-\frac{1}{2}} \cdot \left(-\frac{x}{2} \right) \end{aligned}$$

$\therefore$ Slope of tangent

$$= \frac{-1}{2} \left(1 - \frac{x^2}{4}\right)^{-\frac{1}{2}} \left(\frac{-x}{2}\right) \Big|_{x=\sqrt{2}}$$
$$= -\frac{\sqrt{2}}{4} \left(\frac{1}{2}\right)^{-\frac{1}{2}} = \frac{1}{2}$$

tangent: $\left(\frac{y - \frac{1}{\sqrt{2}}}{x - \sqrt{2}}\right) = \frac{1}{2}$

normal: $\left(\frac{y - \frac{1}{\sqrt{2}}}{x - \sqrt{2}}\right) = -2$

Method (b): (implicit diff.)

$$\frac{x^2}{4} + y^2 = 1 \text{ implicitly}$$

defines a function $y = y(x)$

near $(\sqrt{2}, \frac{1}{\sqrt{2}})$

i.e. $\frac{x^2}{4} + y^2(x) = 1$

$$\frac{x^2}{4} + y^2(x) = 1 \quad \text{for all } x \text{ near } \sqrt{2}$$

$$\frac{d}{dx} \Rightarrow \frac{x}{2} + 2y(x) \cdot y'(x) = 0$$

$$\text{Substitute } x \leftarrow \sqrt{2}, y(x) \leftarrow \frac{1}{\sqrt{2}}$$

$$\Rightarrow y' = \frac{-x}{4y(x)} = -\frac{1}{2}$$

$\Rightarrow$ tangent line, normal line.

Example: $\frac{(x+y)^2}{32} + \frac{(x-y)^2}{8} = 1$

implicitly defines $y = y(x)$
near $(3, 1)$. Find $y'(3)$ and $y''(3)$

Sol: $y'(3)$.

$$\frac{d}{dx} \Rightarrow \frac{(x+y)(1+y')}{16} + \frac{(x-y)(1-y')}{4} = 0$$

$$\left. \begin{matrix} (x, y) \\ = (3, 1) \end{matrix} \right\} \Rightarrow \frac{4(1+y')}{16} + \frac{2(1-y')}{4} = 0, y'(3) = 3$$

$$y''(3): \frac{d}{dx} \left(\frac{(x+y)(1+y)}{16} + \frac{(x-y)(1-y)}{4} = 0 \right)$$

$$\Rightarrow \frac{(1+y')(1+y) + (x+y)y''}{16} + \frac{(1-y')(1-y) + (x-y)y''}{4} = 0$$

Substitute $x=3, y(3)=1, y'(3)=3$

$$\frac{(1+3)^2 + (1+3)y''}{16} + \frac{(1-3)^2 + (1-3)y''}{4} = 0$$

$$1+1 = \frac{y''}{2} - \frac{y''}{4}, \quad y''(3) = 4$$

Example: $x = \tan y, \quad \frac{dy}{dx} = ?$

$$\Rightarrow 1 = (\sec^2 y) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$$

