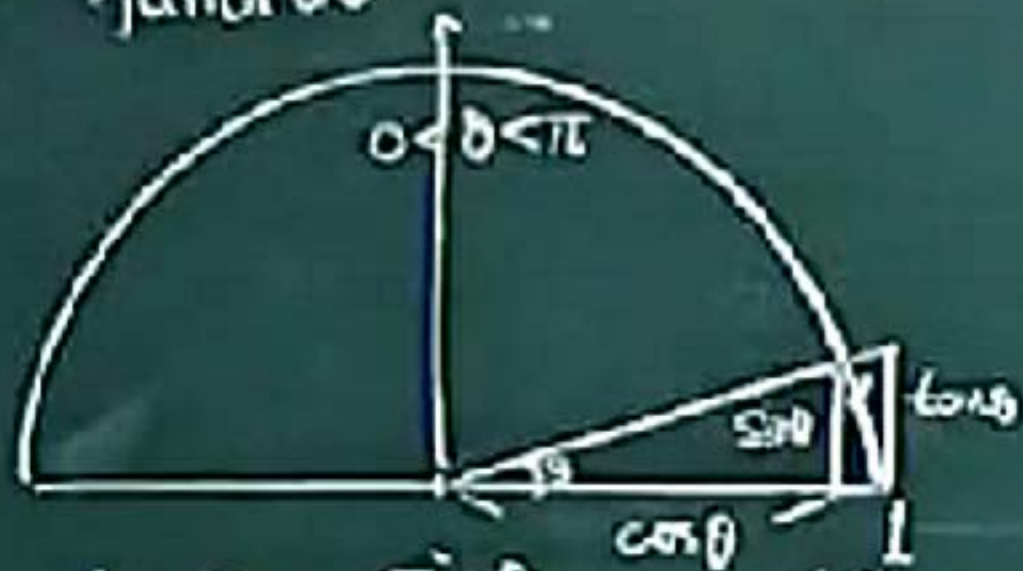


Derivatives of Trigonometric function

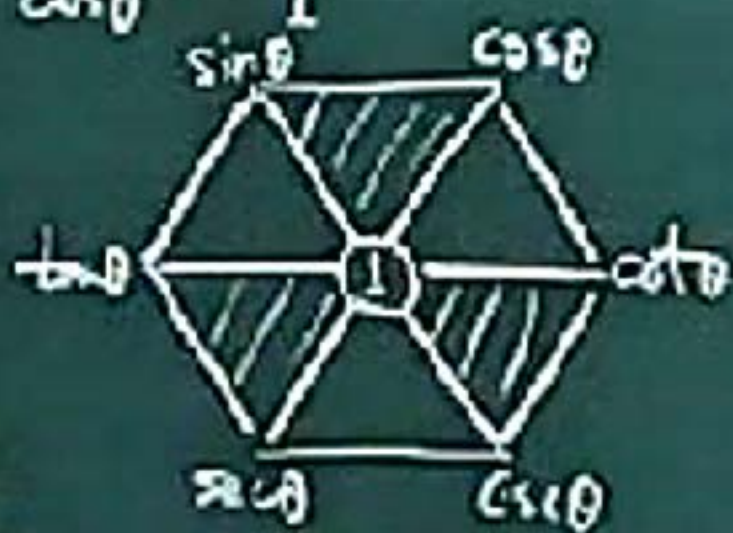


$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

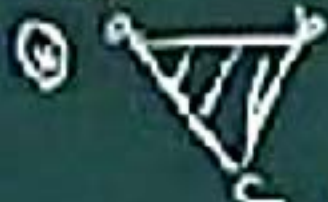


When $-\pi < \theta < 0$

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

Also $\sin(\theta + \pi) = -\sin \theta$
 $\cos(\theta + \pi) = -\cos \theta$



$$a^2 + b^2 = c^2$$



$$p \cdot b = 1$$

We also have

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\cos(2A) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$$

Example: $f(x) = \sin x$, $f'(x) = ?$

Sol: $f'(x) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$

$$= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h \rightarrow 0} \left(\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right)$$

$$= \sin x \underbrace{\lim_{h \rightarrow 0} \frac{-2\sin^2(\frac{h}{2})}{h}}_0 + \cos x \underbrace{\lim_{h \rightarrow 0} \frac{\sin h}{h}}_1$$

$$= \cos x, \text{ i.e., } \underline{\underline{\frac{d}{dx} \sin x = \cos x}}$$

Similarly, one can show that

$$\frac{d}{dx} \cos x = -\sin x$$

Therefore, we also have

$$(\tan x)' = \left(\frac{\sin x}{\cos x} \right)' = \frac{\cos x (\sin x)' - \sin x (\cos x)'}{\cos^2 x} = \sec^2 x$$

$$(\cot x)' = \left(\frac{\cos x}{\sin x} \right)' = \frac{\sin x (\cos x)' - \cos x (\sin x)'}{\sin^2 x} = -\csc^2 x$$

$$(\sec x)' = \left(\frac{1}{\cos x} \right)' = \frac{-1 \cdot (\cos x)'}{\cos^2 x} = \tan x \sec x$$

$$(\csc x)' = \left(\frac{1}{\sin x} \right)' = \frac{-1 \cdot (\sin x)'}{\sin^2 x} = -\cot x \cdot \csc x$$

Example: $f(x) = \frac{(x-1)(x^2-2x)}{x^4}, x \neq 0. f'(x) = ?$

Ans: $f(x) = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}$

$$f'(x) = -x^{-2} + 6x^{-3} - 6x^{-4}$$

Example: $\frac{d}{dx} \left(\frac{\sin x}{x} \right) = ? \quad x \neq 0$

Ans:
$$= \frac{x(\sin x)' - (\sin x)x'}{x^2}$$
$$= \frac{x \cos x - \sin x}{x^2}$$

Example $f(x) = (2-x) \tan^2 x$
 $f'(x) = ?$

Ans: Step 1:

$$(\tan^2 x)' = (\tan x \tan x)'$$

$$= 2 \tan x (\tan x)' = 2 \tan x \sec^2 x$$

Step 2:

$$f'(x) = (2-x)' \tan^2 x + (2-x) (\tan^2 x)'$$
$$= -\tan^2 x + (2-x) \tan x \sec^2 x$$

Chain Rule: $\frac{d}{dx} f(g(x)) = ?$

Example: $\frac{d}{dx} \sin(\tan x) = ?$

Theorem: If $y = f(u)$ is differentiable at $u = g(x)$ and $g(x)$ is diff. at x .

then $f \circ g(x) = f(g(x))$ is differentiable at x .

$$\text{and } \frac{dy}{dx} = \left. \frac{dy}{du} \right|_{u=g(x)} \cdot \frac{du}{dx}$$

$$\left(\text{i.e. } \frac{d}{dx} f(g(x)) = \left. \frac{df(u)}{du} \right|_{u=g(x)} \cdot \frac{d}{dx} g(x) \right) \quad (*)$$

Example: $\frac{d}{dx} \sin(\tan x)$

$$= (\sin u)'_{u=\tan x} \cdot (\tan x)'$$

$$= \cos(\tan x) \cdot \sec^2 x$$

Example: $(\tan^7 x)' = ?$

Ans: Let $f(u) = u^7$
 $g(x) = \tan x$

$$\tan^7 x = f(g(x))$$

$$\left(\text{Remark: } \begin{cases} (f(g(x)))' = \frac{d}{dx} f(g(x)) \\ f'(g(x)) = \frac{d}{du} f(u) \big|_{u=g(x)} \end{cases} \right)$$

$$= \frac{d}{du} u^7 \big|_{u=g(x)} \cdot g'(x)$$

$$= 7(\tan x)^6 \cdot \sec^2 x$$

$$\text{Example } \frac{d}{dx} \sin(x^2 + e^x) \\ = (\cos(x^2 + e^x)) \cdot (2x + e^x)$$

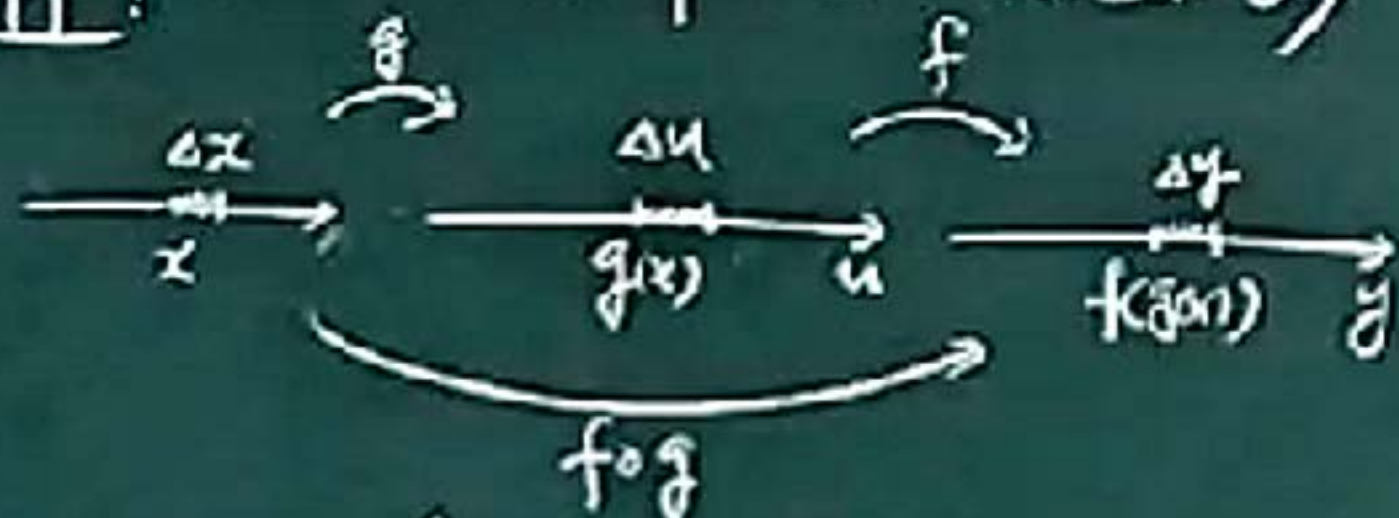
$$\text{Example: } \frac{d}{dx} \sin(\cos(x^2 + 1)) \\ = (\cos(\cos(x^2 + 1))) \cdot ((-\sin(x^2 + 1)) \cdot (2x))$$

$$* \frac{d}{dx} u^n(x) = n u^{n-1}(x) \cdot u'(x)$$

$$* \frac{d}{dx} e^{u(x)} = e^{u(x)} \cdot u'(x)$$

$$* \sin(u(x)) = (\cos(u(x))) \cdot u'(x)$$

pf: (with a minor problem when $\Delta u = 0$)



$$g'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$f'(u) = \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \rightarrow 0} \frac{f(u+\Delta u) - f(u)}{\Delta u} \quad \begin{matrix} u \rightarrow g(x) \\ u = g(x) \end{matrix}$$

$$\frac{d}{dx} f(g(x)) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}$$

$$\text{Since } \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \dots (*)$$

$$\text{Since } \Delta u \rightarrow 0 \text{ when } \Delta x \rightarrow 0, \lim_{\Delta u \rightarrow 0} = \lim_{\Delta x \rightarrow 0}$$

Take $\lim_{\Delta x \rightarrow 0}$ on both sides of $(*) \Rightarrow (*)$ holds.