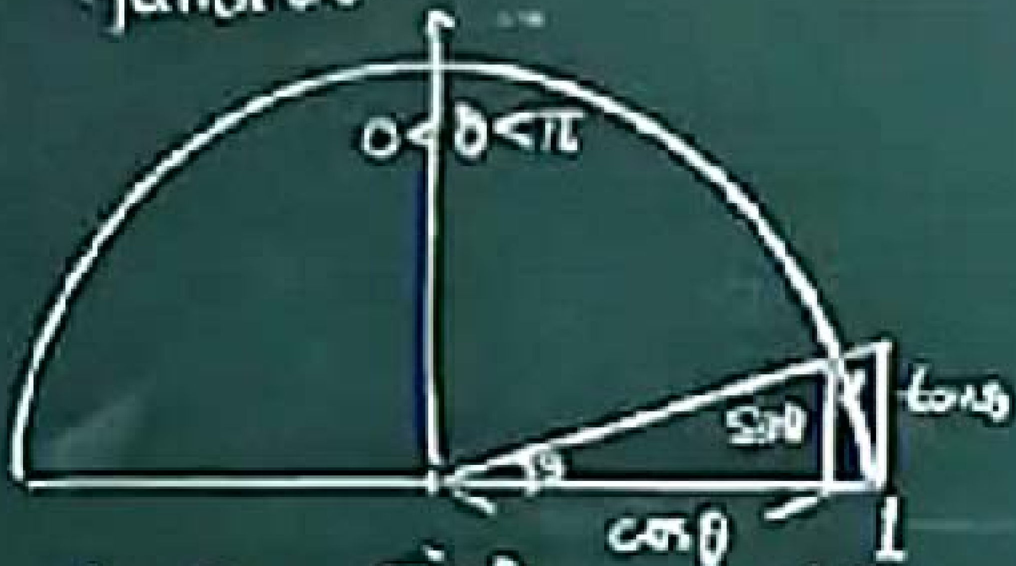
Derivatives of Trigonometric functions

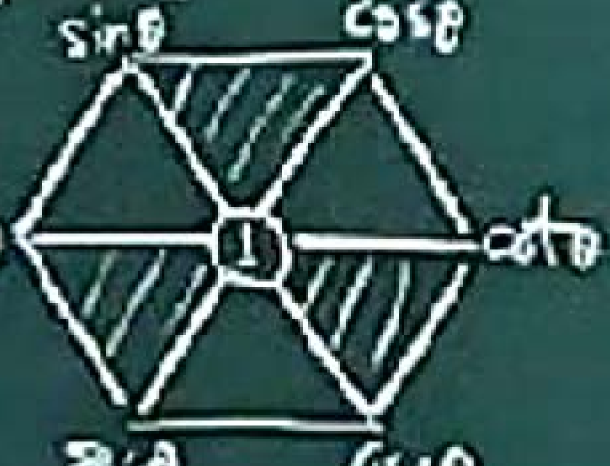


$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$



When $-\pi < \theta < 0$

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

Also $\sin(\theta+\pi) = -\sin\theta$

$\cos(\theta+\pi) = -\cos\theta$

$a^2+b^2=c^2$

$P \cdot Q = 1$

We also have

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\cos(2A) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$$

Example: $f(x) = \sin x$, $f'(x) - ?$

Sol: $f'(x) = \lim_{h \to 0} \dfrac{\sin(x+h) - \sin x}{h}$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h \to 0} \left(\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right)$$

$$= \sin x \underbrace{\lim_{h \to 0} \frac{-2\sin^2(\frac{h}{2})}{h}}_{0} + \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{1}$$

$= \cos x$, i.e. $\underline{\dfrac{d}{dx} \sin x = \cos x}$

Similarly, one can show that

$$\frac{d}{dx}\cos x = -\sin x.$$

Therefore, we also have

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x(\sin x)' - \sin x(\cos x)'}{\cos^2 x} = \sec^2 x.$$

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{\sin x(\cos x)' - \cos x(\sin x)'}{\sin^2 x} = -\csc^2 x$$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{-1\cdot(\cos x)'}{\cos^2 x} = \tan x \sec x$$

$$(\csc x)' = \left(\frac{1}{\sin x}\right)' = \frac{-1\,(\sin x)'}{\sin^2 x} = -\cot x \cdot \csc x$$

Example: $f(x) = \dfrac{(x-1)(x^2-2x)}{x^4}$, $x \neq 0$. $f'(x) = ?$

Ans: $f(x) = \dfrac{x^3-3x^2+2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}$

$$f'(x) = -1x^{-2} + 6x^{-3} - 6x^{-4}$$

Example. $\dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right) = ?, \quad x \neq 0$

Ans: $= \dfrac{x(\sin x)' - (\sin x)x'}{x^2}$

$= \dfrac{x\cos x - \sin x}{x^2}$

Example $f(x) = (2-x)\tan^2 x$

$f'(x) = ?$

Ans: **Step 1:**

$(\tan^2 x)' = (\tan x \tan x)'$

$= 2\tan x (\tan x)' = 2\tan x \sec^2 x$

Step 2:

$f'(x) = (2-x)'\tan^2 x + (2-x)(\tan^2 x)'$

$= -\tan^2 x + (2-x)\underset{\overset{\wedge}{2}}{}\tan x \sec^2 x$

Chain Rule: $\frac{d}{dx} f(g(x)) = ?$

Example: $\frac{d}{dx} \sin(\tan x) = ?$

<u>Theorem</u>: If $y = f(u)$ is differentiable at $u = g(x)$ and $g(x)$ is diff. at x, then $f \circ g(x) = f(g(x))$ is differentiable at x and

$$\frac{dy}{dx} = \left.\frac{dy}{du}\right|_{u=g(x)} \cdot \frac{du}{dx}$$

$$\left(\text{i.e. } \frac{d}{dx} f(g(x)) = \left.\frac{d f(u)}{du}\right|_{u=g(x)} \cdot \frac{d}{dx} g(x) \right) \text{-}(*)$$

Example: $\frac{d}{dx}\sin(\tan x)$

$$= (\sin u)'\Big|_{u=\tan x} \cdot (\tan x)'$$

$$= \cos(\tan x) \cdot \sec^2 x$$

Example: $(\tan^7 x)' = ?$

Ans: Let $f(u) = u^7$

$g(x) = \tan x$

$$\tan^7 x = f(g(x))$$

$$\left(\text{Remark} \begin{cases} (f(g(x)))' = \frac{d}{dx} f(g(x)) \\ f'(g(x)) = \frac{d}{du} f(u)\Big|_{u=g(x)} \end{cases} \right)$$

$$= \frac{d}{du} u^7 \Big|_{u=g(x)} \cdot g'(x)$$

$$= 7(\tan x)^6 \cdot \sec^2 x$$

Example $\frac{d}{dx}\sin(x^2+e^x)$

$$=\left(\cos(x^2+e^x)\right)\cdot(2x+e^x)$$

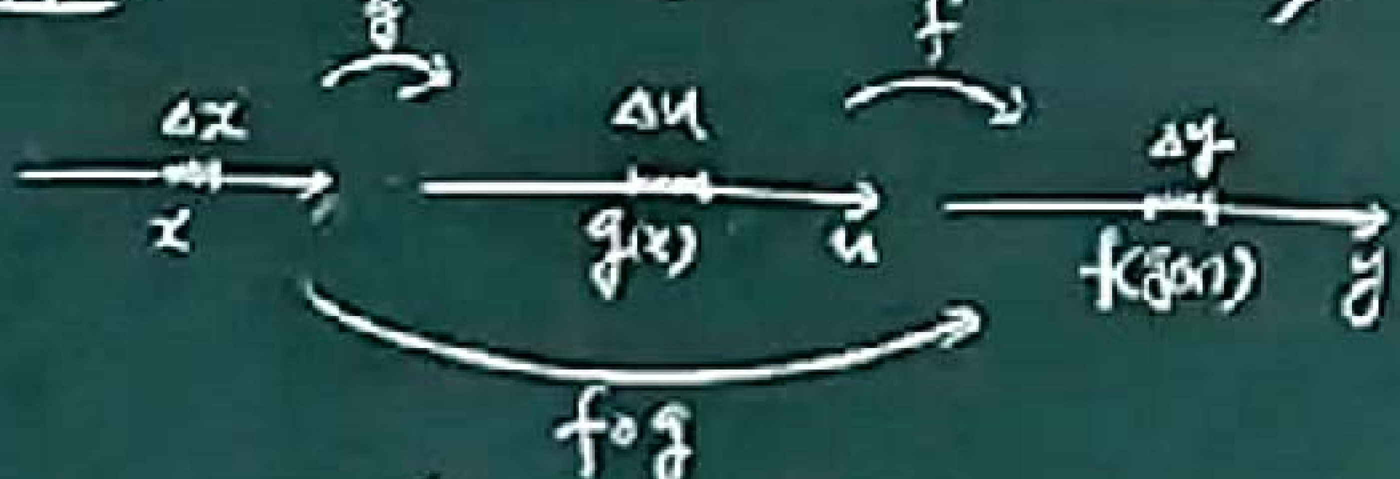
Example $\frac{d}{dx}\sin(\cos(x^2+1))$

$$=\left(\cos(\cos(x^2+1))\right)\cdot\left(\left(-\sin(x^2+1)\right)\cdot(2x+1)\right)$$

* $\frac{d}{dx}u^n(x)=n\,u^{n-1}(x)\cdot u'(x)$
* $\frac{d}{dx}e^{u(x)}=e^{u(x)}\cdot u'(x)$
* $\sin(u(x))=\left(\cos(u(x))\right)\cdot u'(x)$

..

Pf: (with a minor problem when $\Delta u = 0$)



$$g'(x) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$f'(u)\Big|_{u=g(x)} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \to 0} \frac{f(u+\Delta u) - f(u)}{\Delta u}\Big|_{u=g(x)}$$

$$\frac{d}{dx} f(g(x)) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}$$

Since $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$ - - - (**)

Since $\Delta u \to 0$ when $\Delta x \to 0$, $\lim\limits_{\Delta u \to 0} = \lim\limits_{\Delta x \to 0}$

Take $\lim\limits_{\Delta x \to 0}$ on both sides of (**) $\Rightarrow$ (*) holds.