Differentiation Rules

Eg. $\frac{d}{dx}1 = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$(f(x)=1)$ $= \lim_{h\to 0}\frac{1-1}{h} = 0$

Eg. $f(x) = x^2$, $f'(x) = ?$

Ans. $\lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$

$= \lim_{h\to 0}\frac{2xh+h^2}{h} = 2x$

Eg. $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$, $x>0$

$f'(x) = \lim_{z\to x}\frac{\frac{1}{\sqrt{z}} - \frac{1}{\sqrt{x}}}{z-x} \cdot \frac{\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{x}}} = -\frac{1}{2}x^{-\frac{3}{2}}$

Eg: $f(x) = x^n$, $n \in \mathbb{N}$ (positive integer)

$$f'(x) = \lim_{z \to x} \frac{z^n - x^n}{z - x}$$

$$= \lim_{z \to x} z^{n-1} + z^{n-2}x + \cdots + x^{n-1} = nx^{n-1}$$

These are examples of

$$\frac{d}{dx} x^n = n x^{n-1} \quad \text{-----} (*)$$

for $n = 0, 2, \frac{-1}{2}, n \in \mathbb{N}$

Similarly, one can show that (*) holds for $n \in \mathbb{Q}$

ie $n = \frac{q}{p}$, $p, q \in \mathbb{Z}$ $p \neq 0$

Example: $f(x) = x^{\frac{4}{3}}$.

$$f'(x) = \lim_{z \to x} \frac{z^{\frac{4}{3}} - x^{\frac{4}{3}}}{z - x}$$

$$\left((a-b)(a^2+ab+b^2) = a^3 - b^3 \right)$$

$$= \lim_{z \to x} \frac{z^{\frac{4}{3}} - x^{\frac{4}{3}}}{z - x} \, \frac{z^{\frac{8}{3}} + z^{\frac{4}{3}} x^{\frac{4}{3}} + x^{\frac{8}{3}}}{z^{\frac{8}{3}} + z^{\frac{4}{3}} x^{\frac{4}{3}} + x^{\frac{8}{3}}}$$

$$= \lim_{z \to x} \frac{z^4 - x^4}{(z - x)} \, \frac{1}{z^{\frac{8}{3}} + z^{\frac{4}{3}} x^{\frac{4}{3}} + x^{\frac{8}{3}}}$$

$$= \frac{4x^3}{3\,x^{\frac{8}{3}}} = \frac{4}{3} x^{\frac{1}{3}}$$

In fact, $\underline{(*)\text{ holds for } n \in \mathbb{R}}$ (any real number)

We will show it in next section.

$\underline{\text{Eg}}$: $f(x) = x^4 - 2x^2 + 2$.
find all horizontal tangents of $y = f(x)$:

$\underline{\text{Ans}}$: Solve x from $f'(x) = 0$

$$f'(x) = 4x^3 - 4x + 0$$

$$\left(\begin{array}{l}\text{we have used the rule} \\ \frac{d}{dx}\big(u(x)+v(x)\big) = \frac{du(x)}{dx} + \frac{d}{dx}v(x)\end{array}\right)$$

$$\Rightarrow x = 0, \pm 1$$

horizontal tangents at $(0, f(0))$ $(1, f(1))$ and $(-1, f(-1))$ are

$$\frac{y-2}{x-0} = 0 \; ; \; \frac{y-1}{x-1} = 0 \; ; \; \frac{y-1}{x+1} = 0$$

Derivative Rules:

$$\frac{d}{dx}(c \cdot u(x)) = c \frac{d}{dx} u(x) \quad \cdots (1)$$

$c = \text{constant}$

$$\frac{d}{dx}(u(x) \pm v(x)) = \frac{du}{dx} \pm \frac{dv}{dx} \quad \cdots (2)$$

$$\frac{d}{dx}(u(x) \cdot v(x)) = u'v + uv' \quad \cdots (3)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \quad \cdots (4)$$

Pf: (1): exercise

(2): $(u+v)'$

$$= \lim_{h \to 0} \frac{(u(x+h) + v(x+h)) - (u(x) + v(x))}{h}$$

$$= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \to 0} \frac{v(x+h) - v(x)}{h}$$

$= u' + v'$ (Similarly for (2) $-$)

Differentiation Rules

pf of (3).

$$(uv)' = \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\begin{array}{l} u(x+h)v(x+h) - u(x+h)v(x) \\ + u(x+h)v(x) - u(x)v(x) \end{array} \right)}{h}$$

$$= \lim_{h \to 0} u(x+h) \left(\frac{v(x+h) - v(x)}{h} \right)$$

$$+ \lim_{h \to 0} v(x) \left(\frac{u(x+h) - u(x)}{h} \right)$$

$$= u(x)v'(x) + v(x)u'(x)$$

Pf of (4):

$$\frac{u}{v} = u\cdot\left(\frac{1}{v}\right)$$

It remains to evaluate

$$\left(\frac{1}{v}\right)' = \lim_{z\to x} \frac{\frac{1}{v(z)} - \frac{1}{v(x)}}{z-x}$$

$$= \lim_{z\to x} \frac{\left(\frac{v(x)-v(z)}{v(z)\,v(x)}\right)}{z-x}$$

$$= -\lim_{z\to x} \left(\frac{v(z)-v(x)}{z-x}\right)\frac{1}{v(z)\,v(x)}$$

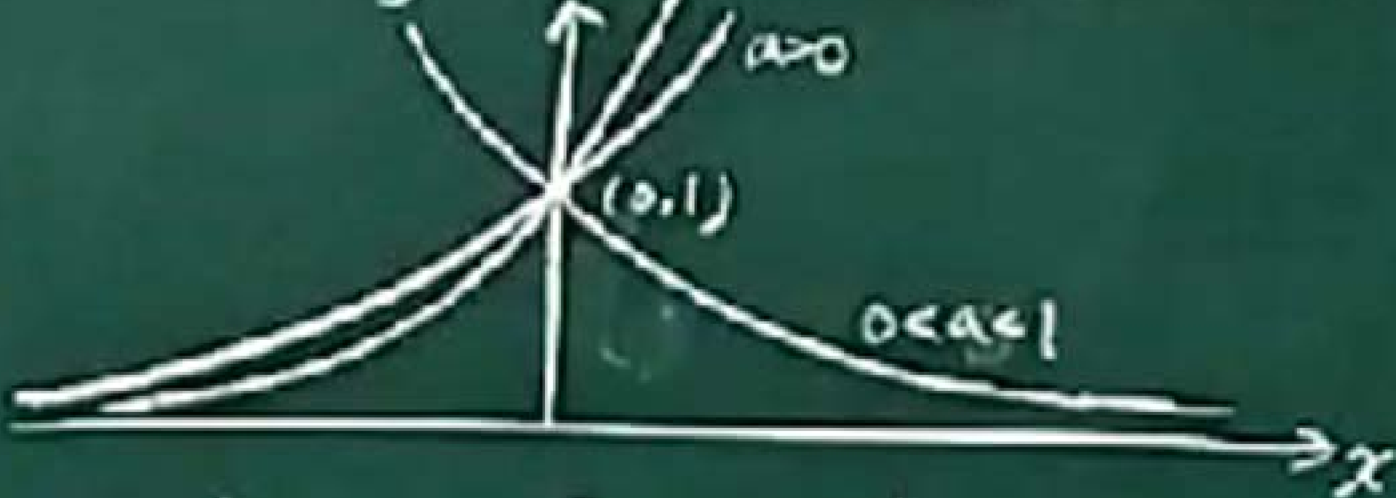
$$= -\frac{v'}{v^2}$$

From (3) $\Rightarrow \left(\frac{u}{v}\right)' = u\left(\frac{1}{v}\right)' + u'\left(\frac{1}{v}\right)$

$$= \frac{vu' - uv'}{v^2}$$

Exponential functions

$$y = a^x, \quad x \in \mathbb{R}, \quad a > 0$$



$$\frac{d}{dx} a^x = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= \Big(\underbrace{\lim_{h \to 0} \frac{a^h - 1}{h}}_{\parallel} \Big) \cdot a^x$$

$$= g(a) \cdot a^x$$

Here $g(a)=\lim_{h\to 0}\frac{a^h-1}{h}$ (if exists)

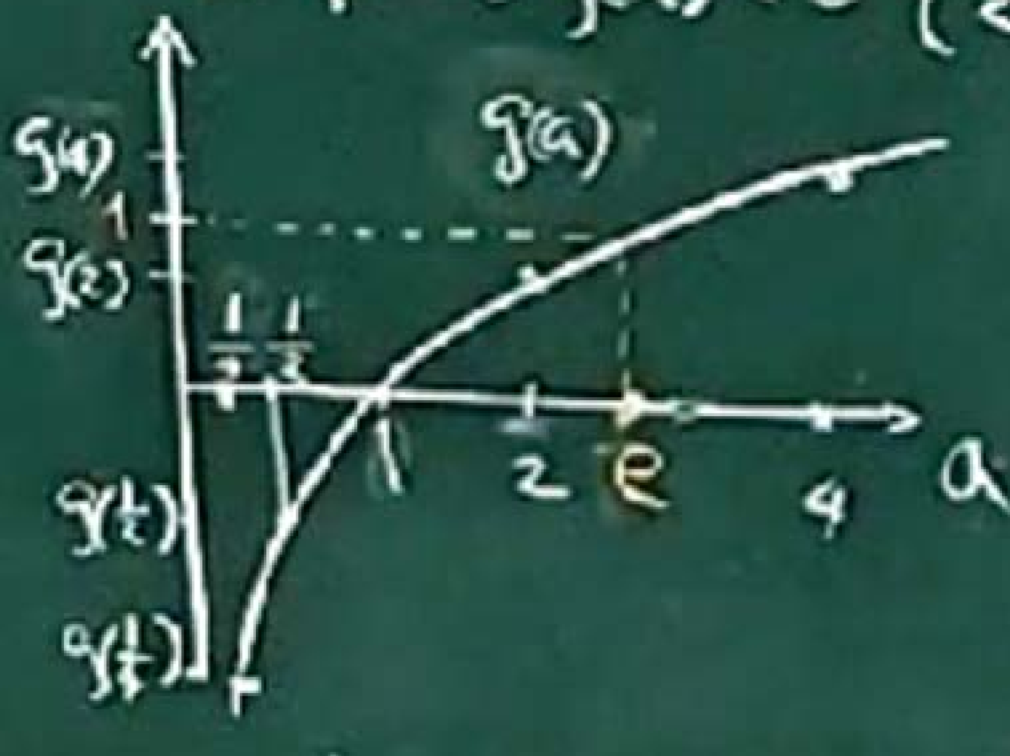
$$g(1)=\lim_{h\to 0}\frac{1^h-1}{h}=0$$

$$g(a^2)=\lim_{h\to 0}\frac{a^{2h}-1}{h}\cdot\frac{2}{2}$$

$$=\lim_{h\to 0}\frac{a^{2h}-1}{2h}\cdot 2=2g(a)$$

$a>1 \Longrightarrow g(a)\geq 0$ (In fact, it exists and >0)

$a<1 \Longrightarrow g(a)\leq 0$ (<0



Summary:

(1) $\frac{d}{dx}a^x = \left(\lim_{h\to\infty}\frac{a^h-1}{h}\right)\cdot a^x$

(2) Let $e \simeq 2.71828\cdots$

satisfy $\left(\lim_{h\to 0}\frac{e^h-1}{h}\right) = 1$

i.e. $\frac{d}{dx}e^x = e^x$

<u>Example</u> $f(x) = e^{-x}$

$f'(x) = ?$

$f(x) = \frac{1}{e^x}$, from proof of (4) above

$f'(x) = \frac{-(e^x)'}{(e^x)^2} = \frac{-e^x}{e^{2x}}$

$= -e^{-x}$