Def: Derivative of f at x_0

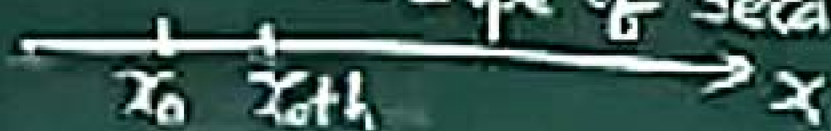
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$=$ slope of tangent ....

$y = f(x)$

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{\Delta y}{\Delta x}$$

$=$ slope of secant



$h \to 0$ $(\Delta x \to 0)$

Slope of secant $\to$ slope of tangent

(when $x =$ time)

$=$ rate of change of y

$f'(x_0)$: derivative of f at x_0

$f'(x)$: derivative of f as a function.

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

Notation: $\frac{d}{dx} f(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$

$D_x f$, $\left.\frac{d}{dx} f(x)\right|_{x=x_0}$, $\left.\frac{dy}{dx}\right|_{x=x_0}$, ...

Example: $f(x) = \frac{1}{x}$, $x \neq 0$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}$$

Remark. f is a differentiable function if $f'(x)$ exists for any x in its domain.

Remark: If f is defined on $[a,b]$. Then f is right (left) differentiable at $x = a$ (b) if

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

$$\left(\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \right)$$

exists.

Example: $f(x)=\sqrt{x},\quad x\geq 0$

$f'(x)=?$

<u>Ans.</u> if $x>0$

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{(\cancel{\sqrt{x+h}-\sqrt{x}})(\sqrt{x+h}+\sqrt{x})}{\cancel{h}\,(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2}\frac{1}{\sqrt{x}}$$

If $x=0$

$$\lim_{h\to 0^+}\frac{\sqrt{h+0}-\sqrt{0}}{h}=\lim_{h\to 0^+}\frac{1}{\sqrt{h}}=+\infty$$

Relation between continuity and differentiability

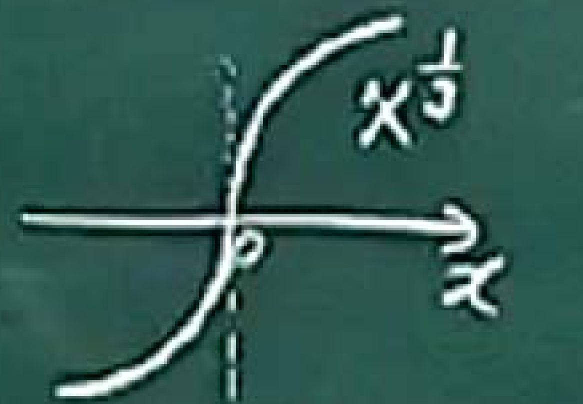
$f(x)$ is cont. at x_0 $\not\Rightarrow$ $f(x)$ is diff. at x_0

$\Leftarrow$

$\not\Rightarrow$:

(corner) ,

(cusp)



(vertical tangent)

$\Leftarrow$:

Theorem: If f has a derivative at c, then f is continuous at c.

Pf: (cont: $\Delta y \to 0$ as $\Delta x \to 0$)

(diff. $\frac{\Delta y}{\Delta x} \to f'(c)$ as $\Delta x \to 0$)

$$f(c+h) = f(c) + \frac{f(c+h)-f(c)}{h}\cdot h, \quad h \neq 0$$

$$\therefore \lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c) + \left(\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\right)\left(\lim_{h\to 0} h\right) = f(c)$$