<u>Def</u> Suppose $f(x)$ is defined on $[a, b]$.

If $c \in (a, b)$, then f
$c \in (a, b]$
$c \in [a, b)$

is continuous at c, if
left continuous
right continuous

$$\lim_{x \to c} f(x) = f(c)$$
$x \to c^-$
$x \to c^+$

i.e. For any $\varepsilon > 0$, there exists a $\delta > 0$, such that

"(or) $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$"

$c - \delta < x \leq c$

$c \leq x < c + \delta$

We say that f is a continuous function if f is continuous at every point of its domain.

Basic properties of continuous functions,
See Section 2.5, Thm 8.

<u>Example</u>. $\lim_{x \to 0} \sqrt{x+1}\, 2^{\sin x}$

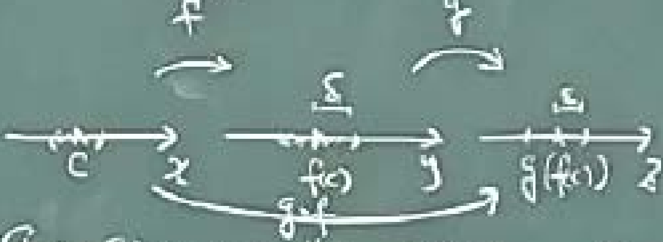
<u>Sol</u>. $f(x) = \sqrt{x+1}\, 2^{\sin x}$ is a continuous function near $x=0$ (see composite of continuous functions below)

$\therefore$ Ans $= f(0) = 1$

Theorem: [Composite of continuous functions]

If f is continuous at c, g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

(i.e. $\lim_{x \to c} g(f(x)) = g(f(c))$)



$g(y)$: given $\varepsilon > 0$, there exists $\delta_1 > 0$, such that

"$|y - f(c)| < \delta_1 \Longrightarrow |g(y) - g(f(c))| < \varepsilon$" ...(1)

$f(x)$: given $\delta_1 > 0$, there exists $\delta > 0$, such that

"$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1$" ...(2)

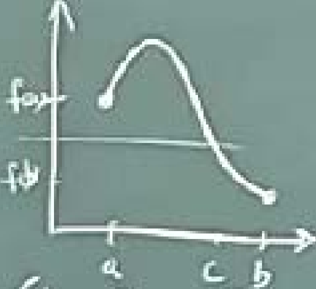
From (1), (2) $\Rightarrow$ "$|x - c| < \delta \Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$"

Remark: Similarly,

if g is continuous at b

and $\lim\limits_{x\to c} f(x) = b$, then

$$\lim_{x\to c} g(f(x)) = g(b) = g\left(\lim_{x\to c} f(x)\right)$$



(typical case)

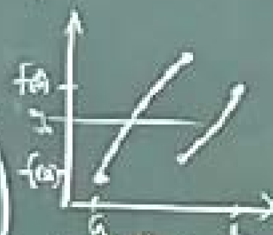
Theorem [Intermediate Value Theorem]

If f is continuous on $[a,b]$

$\Rightarrow$ f takes any value between $f(a)$ and $f(b)$.

(i.e. if y_0 is any value between $f(a)$ and $f(b)$, then $\exists\, c \in [a,b]$ such that $f(c) = y_0$.)

(Continuity is necessary)



(✱)

Example Show that $f(x)=x^3-x-1=0$ has a root in $(1,2)$.

Sol: $f(1)=-1$, $f(2)=5$

$\xrightarrow{\text{IVT}} \exists x\in(1,2),\ f(x)=0$

Example: Show that

$\sqrt{2x+5}=4-x^2$ has a solution

Sol: Let $f(x)=\sqrt{2x+5}+x^2-4$

$f(0)=\sqrt{5}-4<0$, $f(2)=\sqrt{9}>0$

$\text{IVT}\Rightarrow \exists x\in(0,2)\ f(x)=0$

Section 2.6 Limits involving infinity

[Skip asymptotes, will visit later]

Recall: $\lim_{x \to c^-} f(x) = L$

"For any $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$c - \delta < x < c \Rightarrow |f(x) - L| < \varepsilon$"

Def $\lim_{x \to \infty} f(x) = L$ ($x \to -\infty$)

For any $\varepsilon > 0$, there exists $M \in \mathbb{R}$ ($N \in \mathbb{R}$) such that

"$x > M$ ($x < N$) $\Rightarrow |f(x) - L| < \varepsilon$"

Example use the precise definition to show $\lim_{x \to \infty} \frac{1}{x} = 0$

Sol. For any $\varepsilon > 0$

(find x such that)

$$\left| \frac{1}{x} - 0 \right| < \varepsilon$$

$$\Longleftrightarrow \begin{cases} 0 < \frac{1}{x} < \varepsilon \\ \text{or } 0 > \frac{1}{x} > -\varepsilon \end{cases}$$

$$\Longleftarrow \quad 0 < \frac{1}{x} < \varepsilon$$

$$\Longleftrightarrow \quad x > \frac{1}{\varepsilon}$$

That is, take $M = \frac{1}{\varepsilon}$ will do.

Def $\lim_{x \to c} f(x) = \infty$ $(-\infty)$

$$\left(\begin{array}{c} |f(x) - L| < \varepsilon \\ (f(x) \text{ is close to } L) \\ \Updownarrow \\ (f(x) \text{ is close to } \infty) \\ f(x) > B \end{array} \right)$$

For any $B > 0$, there exists $\delta > 0$, such that

"$0 < |x - c| < \delta \Rightarrow f(x) > B$"

$(f(x) < -B)$