One sided limit

Recall $\lim_{x \to c^{\pm}} f(x) = L$

Suppose $f(x)$ is defined on

$(c-a, c) \cup (c, c+a)$, $a > 0$.

$\left((c-a, c), (c, c+a) \right)$

If for any $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$0 < |x-c| < \delta \Longrightarrow |f(x) - L| < \varepsilon$$

$$(c < x < c+a)$$

$$(c-a < x < c)$$

Example · $\lim_{x \to 0^+} \sin\frac{1}{x}$ exists?

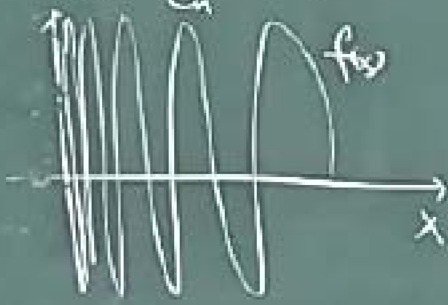
Ans: No. Let $a_n = \frac{1}{n\pi}$

$b_n = \frac{1}{(2n+\frac{1}{2})\pi}$

$C_n = \frac{1}{(2n-\frac{1}{2})\pi}$

$n = 1, 2, 3, \cdots$

$$\Rightarrow \begin{cases} \sin\frac{1}{a_n} = 0 \\ \sin\frac{1}{b_n} = 1 \\ \sin\frac{1}{c_n} = -1 \end{cases}$$



Since in any interval $(0,\delta)$, we can always find $a_n, b_n, c_n \in (0,\delta)$ with $\begin{cases} f(a_n) = 0 \\ f(b_n) = 1 \\ f(c_n) = -1 \end{cases}$

i.e. values of $f(x)$ are not all near any L when $x \in (0,\delta)$

Remark: How to prove

$$\lim_{x \to x} f(x) \neq L$$

Recall $\lim_{x \to c} f(x) = L$

(S_1) For any $\varepsilon > 0$.

there exists $\delta > 0$ such that

"$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$" (S_2)

(S_3)

(S_1) is true $\Leftrightarrow$ For any $\varepsilon > 0$, (S_2) is true

(S_1) is false $\Leftrightarrow$ There exists an $\varepsilon > 0$ such that (S_2) is false

(S_2) is true $\Leftrightarrow$ there exists a $\delta > 0$, such that (S_3) is true.

(S_2) is false $\Leftrightarrow$ For any $\delta > 0$, (S_3) is false.

(S_3) is true $\Leftrightarrow$ '$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$'

(S_3) is false $\Leftrightarrow$ there exists an x, $0 < |x - c| < \delta$ such that $|f(x) - L| \geq \varepsilon$

In summary $\lim_{x \to c} f(x) = L$ is false

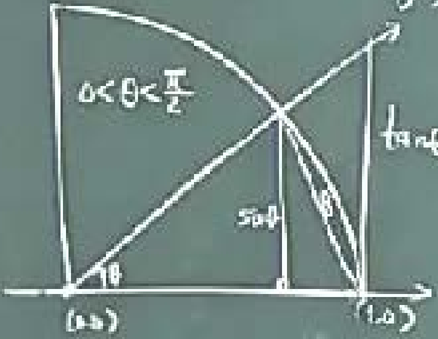
There exists an $\varepsilon > 0$ such that for any $\delta > 0$, there exists an x with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$.

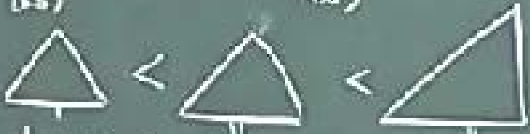
In previous example $\left(\lim_{x \to 0^+} \sin \frac{1}{x} \right)$

It suffices to take $\varepsilon = \frac{1}{2}$ (it will work for any L)

§ Limits involving $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$.

<u>Sol</u>: we start with $\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}$





$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$\left(\times\frac{2}{\sin\theta}\right) \Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$1 > \frac{\sin\theta}{\theta} > \cos\theta, \quad 0<\theta<\frac{\pi}{2}$$

Since $\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$

$$\Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta$$

on $\left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$

Since $\lim\limits_{\theta\to 0} = 1 = \lim\limits_{\theta\to 0} \cos\theta$

Sandwich Thm

$$\Rightarrow \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Example: $\lim\limits_{t\to 0} \dfrac{\cos(t)-1}{t} = ?$

$$= \lim_{t\to 0} \frac{-2\sin^2\left(\frac{t}{2}\right)}{t}$$

$$= -\lim_{t\to 0} \frac{\sin\left(\frac{t}{2}\right)}{\frac{t}{2}} \left(\sin\frac{t}{2}\right) = 0$$

Example: $\lim_{t \to 0} \frac{\tan t}{t}$

$$= \lim_{t \to 0} \frac{\sin t}{t} \frac{1}{\cos t} = 1$$

Example: $\lim_{t \to 0} \frac{\sin(\sin t)}{t}$

$$= \lim_{t \to 0} \frac{\sin(\sin t)}{\sin t} \frac{\sin t}{t}$$

$$= \lim_{\sin t \to 0} \frac{\sin(\sin t)}{\sin t} \lim_{t \to 0} \frac{\sin t}{t}$$

$$= 1$$

Continuity

Definition: Suppose $f(x)$ is defined on $(c-a, c+a)$, $a>0$
Then $f(x)$ is continuous at c if $\lim_{x \to c} f(x) = f(c)$ ... (*)

Remark (*) means
(1): $f(c)$ exists
(2): $\lim_{x \to c} f(x)$ exists
(3): (1) = (2)

Alternatively, we can define
"$f(x)$ is continuous at c"
using $\varepsilon-\delta$ as follows:

Suppose $f(x)$ is defined
on $(c-a, c+a)$, $a>0$.
Then $f(x)$ is continuous at c
if: For any $\varepsilon>0$, there
exists a $\delta>0$, such that
" $|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$ "
$(\forall x)$