~~Formal~~ Precise definition of $\lim_{x \to c} f(x) = L$

Informal definition

(*). $f(x)$ is close to $L$ whenever $x$ is close to $c$.

problem: what is "close"?

(**) $f(x)$ is arbitrarily close to $L$ (as close to $L$ as we like) for all $x$ sufficiently close to $c$.

problem: "arbitrarily", "suffiently" are not precise.

## Precise definition.

Suppose $f(x)$ is defined on $(c-a, c) \cup (c, c+a)$

$a > 0$. $[(c-a, c+a) \setminus \{a\}]$

Then $\lim_{x \to c} f(x) = L$

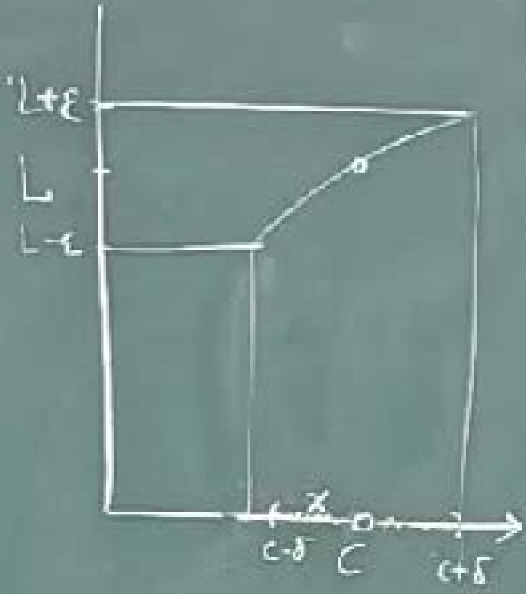
if, for every [any] (number) $\varepsilon > 0$, there exists

a (corresponding) (number) $\delta > 0$, such that

(for all $x$)

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \quad (*)$$

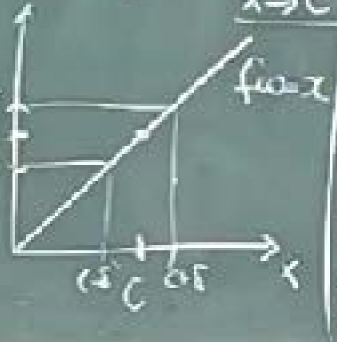
Remark: It suffices to find $\delta > 0$ for all $\varepsilon \in (0, b]$, $b > 0$.




How to prove $\lim_{x\to c} f(x)=L$ ?

"Describe": for any $\varepsilon>0$, how to find a corresponding $\delta>0$ satisfying (☆)

<u>Example</u>. $\lim_{x\to c} x = c$




<u>Ans</u>: take $\delta=\varepsilon$

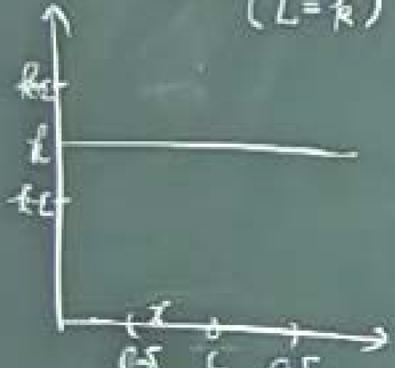
$0<|x-c|<\varepsilon=\delta$

$\Rightarrow |\underbrace{x}_{f(x)}-\underbrace{c}_{L}|<\varepsilon$

Example ($f(x)=k$)

$$\lim_{x\to c} k = k$$

($L=k$)




Ans. For any $\varepsilon>0$ take any $\delta>0$.

Since $0<|x-c|<\delta \Rightarrow |k-k|<\varepsilon$

yes

Example.

In "$\lim_{x\to 5}\sqrt{x-1}=2$", find $\delta$ for $\varepsilon=1$.

Ans. $\left|\sqrt{x-1}-2\right|<1$

$\Leftrightarrow -1<\sqrt{x-1}-2<1$

$\Leftrightarrow 1<\sqrt{x-1}<3$

$\Leftrightarrow 1<x-1<9$

$\Leftrightarrow 2<x<10$

$\Leftrightarrow -3<x-5<7$

$\xLeftarrow{\text{take } \delta=3} 0<|x-5|<3$

Example: $f(x)=\begin{cases} x^2, & x\neq 2 \\ 1, & x=2 \end{cases}$

Prove that $\lim\limits_{x\to 2} f(x) = 4$

pf: For any $\varepsilon>0$, $\left(\begin{matrix}\text{assume} \\ \varepsilon<4\end{matrix}\right)$

$$|x^2-4|<\varepsilon, \quad \underline{\underline{(x\neq 2)}}$$

$$\Leftrightarrow -\varepsilon < x^2-4 < \varepsilon$$

$$\Leftrightarrow 4-\varepsilon < x^2 < 4+\varepsilon$$

$$\Leftrightarrow \begin{cases} \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon} \\ \text{or} \\ -\sqrt{4+\varepsilon} < x < -\sqrt{4-\varepsilon} \end{cases}$$

$$\Leftarrow \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$

$$\Leftrightarrow \underbrace{\sqrt{4-\varepsilon}-2}_{\text{negative}} < x-2 < \underbrace{\sqrt{4+\varepsilon}-2}_{\text{positive}}$$

$\left[\begin{matrix}\text{Need } \delta>0 \text{ to satisfy} \\ (-\delta, \delta) \subseteq (\sqrt{4-\varepsilon}-2, \sqrt{4+\varepsilon}-2)\end{matrix}\right]$ (◎)

Then

$$\Leftarrow -\delta < x-2 < \delta$$

$$\Leftarrow 0<|x-2|<\delta$$

$$\left[\begin{matrix} (◎) \text{ requires} \\ \begin{cases} \sqrt{4-\varepsilon}-2 \le -\delta \\ \delta \le \sqrt{4+\varepsilon}-2 \end{cases} \\ \Leftrightarrow \begin{cases} \delta \le 2-\sqrt{4-\varepsilon} \\ \delta \le \sqrt{4+\varepsilon}-2 \end{cases} \end{matrix}\right]$$

Take $\delta = \min(2-\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}-2)$

Example If $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$

Prove that $\lim_{x\to c}(f(x)+g(x)) = L+M$.

pf For any $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$, such that

$$0<|x-c|<\delta_1 \Rightarrow -\frac{\varepsilon}{2} < f(x)-L < \frac{\varepsilon}{2}$$

$$0<|x-c|<\delta_2 \Rightarrow -\frac{\varepsilon}{2} < g(x)-M < \frac{\varepsilon}{2}$$

take $\delta = \min(\delta_1, \delta_2)$

$$0<|x-c|<\delta \Rightarrow \begin{cases} -\frac{\varepsilon}{2} < f(x)-L < \frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} < g(x)-M < \frac{\varepsilon}{2} \end{cases}$$

$$\Rightarrow -\varepsilon < f(x)-L+g(x)-M < \varepsilon$$

$$\Leftrightarrow |f(x)+g(x)-(L+M)| < \varepsilon$$