Chap 2 Limit and Continuity

Review of "Limit"

Informal definition:

Suppose $f(x)$ is defined on $(x_0-\delta, x_0) \cup (x_0, x_0+\delta)$ for some $\delta > 0$. We say that

$$\lim_{x \to x_0} f(x) = L$$

if $f(x)$ approaches L when x approaches x_0

Note: $f(x_0)$ is not relevant.

Example 3 (a) $f(x) = x$, then $\lim\limits_{x \to x_0} f(x) = x_0$

(b) $f(x) = k$, then $\lim\limits_{x \to x_0} f(x) = k$

Example $\lim\limits_{x \to 0} f(x)$ does not exist.

(i) $f(x) = \dfrac{x}{|x|}$, $x \neq 0$, (jump at $x = 0$)

(ii) $f(x) = \dfrac{1}{x}$ or $\dfrac{1}{x^2}$, $x \neq 0$, $\left(\begin{array}{l}\text{diverges to } \pm\infty; \\ \text{unbounded near } x=0\end{array}\right)$

(iii) $f(x) = \sin\dfrac{1}{x}$, $x \neq 0$ (oscillation near $x = 0$)

§ Basic limits

Theorem 1 (Limit laws)

If $\lim\limits_{x\to c} f(x) = L$, $\lim\limits_{x\to c} g(x) = M$

① ②: $\lim\limits_{x\to c}\left(f(x) \pm g(x)\right) = L \pm M$

③: $\lim\limits_{x\to c}\left(k f(x)\right) = k\left(\lim\limits_{x\to c} f(x)\right)$

④: $\lim\limits_{x\to c}\left(f(x)\cdot g(x)\right) = L \cdot M$

⑤ $\lim\limits_{x\to c}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{L}{M}$; if $M \neq 0$

$$\left.\begin{array}{l} ⑥\ \lim\limits_{x\to c}\left(f(x)^n\right) = \left(\lim\limits_{x\to c} f(x)\right)^n \\ ⑦\ \lim\limits_{x\to c}\left(f(x)^{\frac{1}{n}}\right) = L^{\frac{1}{n}} \left(\begin{array}{l}\text{assume } L>0 \\ \text{if } n \text{ is even}\end{array}\right) \end{array}\right\} \begin{array}{l} n \text{ is a} \\ \text{positive} \\ \text{integer} \end{array}$$

Example 5

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3) = c^3 + 4c^2 - 3$ (Limits of Polynomials)

(b) $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{c^4 + c^2 - 1}{c^2 + 5}$ (Limits of rational functions)

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{13}$

§ Limits involving quotient

$$\frac{\text{"}0\text{"}}{0} \quad \text{or} \quad \frac{\text{"}\neq 0\text{"}}{0}$$

<u>Example 7:</u> $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$

$x^2-x=0$ when $x=1$

$x^2+x-2=0$ when $x=1$

$\Rightarrow \quad x^2-x=(x-1)\cdot x$

$x^2+x-2=(x-1)(x+2)$

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to 1}\frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}x}=3$$

<u>Example 8</u>: $$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$$

$$=\lim_{x\to 0}\frac{\left(\sqrt{x^2+100}-10\right)}{x^2}\frac{\left(\sqrt{x^2+100}+10\right)}{\left(\sqrt{x^2+100}+10\right)}$$

$$(a-b)(a+b)=a^2-b^2$$

$$=\lim_{x\to 0}\frac{\cancel{x^2}}{\cancel{x^2}\left(\sqrt{x^2+100}+10\right)}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}=\frac{1}{20}$$

Similarly $a^3-b^3=(a-b)(a^2+ab+b^2)$

can be applied when $\sqrt[3]{\cdots}$ appears

Example $\lim\limits_{x\to 1} \dfrac{x^2-1}{(x-1)^2}$

$= \lim\limits_{x\to 1} \dfrac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-1)}$ does not exist

§ Sandwich Theorem

Theorem 4 Suppose $g(x) \le f(x) \le h(x)$ on $(c-\delta, c) \cup (c, c+\delta)$, $\delta > 0$.

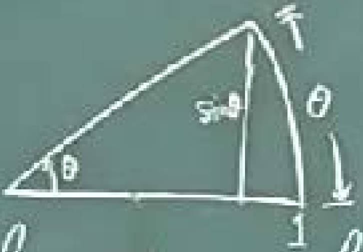
If $\lim\limits_{x\to c} g(x) = L = \lim\limits_{x\to c} h(x)$

Then $\lim\limits_{x\to c} f(x) = L$

Example: $\lim_{\theta \to 0} \sin\theta = ?$

Since $-|\theta| < \sin\theta < |\theta|$

$(\theta > 0)$



$$\lim_{\theta \to 0} -|\theta| = 0 = \lim_{\theta \to 0} |\theta|$$

$$\xRightarrow{\text{Thm } 4} \lim_{\theta \to 0} \sin\theta = 0.$$

Example: $\lim_{x \to 0} \left(x \sin \frac{1}{x} \right)$

Since $-1 \leq \sin \frac{1}{x} \leq 1$

$\Rightarrow -|x| \leq x \sin \frac{1}{x} \leq |x|$

$\lim_{x \to 0} -|x| = 0 = \lim_{x \to 0} |x|$

$\Rightarrow \lim_{x \to 0} x \sin \frac{1}{x} = 0$