A High-Order Kernel-Free Boundary Integral Method for the Biharmonic Equation on Irregular Domains

Yaning Xie, Wenjun Ying & Wei-Cheng Wang

Journal of Scientific Computing

ISSN 0885-7474
Volume 80
Number 3

J Sci Comput (2019) 80:1681-1699
DOI 10.1007/s10915-019-01000-6

Springer

Journal of Scientific Computing (2019) 80:1681–1699
https://doi.org/10.1007/s10915-019-01000-6

A High-Order Kernel-Free Boundary Integral Method for the Biharmonic Equation on Irregular Domains

Yaning Xie[1] · Wenjun Ying[2] · Wei-Cheng Wang[3]

Received: 26 November 2018 / Revised: 13 May 2019 / Accepted: 23 June 2019 / Published online: 5 July 2019

Abstract

This work proposes second-order and fourth-order versions of a Cartesian grid based kernel-free boundary integral (KFBI) method for the biharmonic equation on both bounded irregular domains and singly periodic irregular domains. It is further development of the previous KFBI method for second-order elliptic PDEs. It reformulates boundary value problems of the fourth-order PDE as boundary integral equations of the first kind but the solution never needs to know the fundamental solution or Green's function of the elliptic operator. Evaluation of boundary or volume integrals in the solution of boundary integral equations is made by solving equivalent interface problems on Cartesian grids with standard finite difference methods and fast Fourier transform based solvers. The work decomposes the biharmonic equation into two Poisson equations. It assumes the solution to one Poisson equation, which has no boundary conditions, as the sum of a volume integral with a double layer boundary integral, and applies Green's third identity to derive a scalar boundary integral equation from the other Poisson equation that are subject to two boundary conditions. In the solution of the scalar boundary integral equation, each volume or boundary integral is evaluated with the KFBI method. Numerical examples are presented to demonstrate the solution accuracy and algorithm efficiency. A remarkable point of the work is that the nine-point compact difference scheme in dealing with each split second-order elliptic interface problem on irregular domains yields fourth-order accurate solution for the biharmonic equation.

Keywords Biharmonic equation · Interface problem · Boundary integral method · Cartesian grid method · Nine-point compact finite difference scheme

1 Introduction

The biharmonic equation has important applications in fluid mechanics [36,41,69], linear elasticity [17,68,75] and the theory of thin plates [43,62,79]. The unknown in the biharmonic equation represents the stream function of an incompressible two-dimensional creeping flow

---

Research of the second author is supported by the Science Challenge Project of China under Grant No. TZ2016002 and the National Natural Science Foundation of China under Grant DMS-11771290. Research of the second author was also supported by the Young Thousand Talents Program of China.

Extended author information available on the last page of the article

in fluid mechanics, the Airy stress function in linear elasticity or the vertical displacement due to an external force in the theory of thin plates. Efficient, accurate and stable numerical algorithms for the biharmonic equation remain active research topics in the numerical analysis and scientific computing community over the past decades. There are three major classes of numerical methods for the biharmonic equation in the literature.

Finite element method (FEM) is one of the most widely used techniques for the biharmonic equation. It is based on some variational formulations of the equation and suitable for problems on irregular domains, which are often partitioned into unstructured triangular or tetrahedral grids. Standard FEMs for the biharmonic equation include conforming FEM [3,20], non-conforming FEM [1,4,59,64] as well as mixed FEM [10,16,21,27,34,70]. Conforming FEMs are based on high regularity assumption on the solution and have high order smoothness requirement on finite elements. Non-conforming and mixed FEMs also ask for sophisticated and special finite elements. Usually discrete finite element systems for the biharmonic equation have large condition numbers, which make efficient solution of the resulting equations a problem itself. Recent literature on FEMs for the biharmonic equation includes interior penalty Galerkin FEM [9,78], discontinuous Galerkin FEM [22] and weak Galerkin FEMs [65,66,86]. With FEMs, high-order accuracy may not be easy to obtain and computational cost could be relatively large in general.

Finite difference method (FDM) is another widely used technique for the biharmonic equation. It works with Cartesian grids and is especially good for problems on regular domains such as a rectangle or a union of rectangles [7,77]. Standard FDMs for the biharmonic equation have two approaches: direct methods and indirect methods. The first one directly discretizes the biharmonic operator by a 13-point or 25-point finite difference stencil [23]. The resulting linear system by this approach is ill-conditioned [24,30,37] and requires suitable preconditioning [80] or matrix decomposition technique [7]. The second one is a splitting method, which introduces an intermediate variable and decomposes the fourth-order elliptic equation into two second-order equations [2,28,29]. On regular domains or circular domains, the latter approach yields linear equations that can be efficiently solved by fast Poisson solvers [11,39,58]. The FDM [58] for the biharmonic equation on circular domains solves the problem in polar coordinates. On general irregular domains, finite difference equations should be modified at irregular grid nodes, at which finite difference stencils go across the boundary of the irregular domain. A successful example is the immersed interface method [14], where the two split Poisson equations are solved by the immersed interface method with augmented variables [60]. The method [14] for the biharmonic equation on irregular domains yields second-order accurate results and takes advantages of fast Poisson solvers on Cartesian grids.

Boundary integral equation methods (BIEM), including boundary integral method (BIM) and boundary element method (BEM), are the third class of techniques for the biharmonic equation. For the biharmonic equation with a homogeneous right hand side, the methods transform the problem on a domain into one defined only on the domain boundary. Dimension reduction is a major advantage of BIEMs, which avoids generation of unstructured grids for irregular domains. Standard BIEMs for the biharmonic equation also have two subclasses: direct methods and indirect methods. The direct methods (e.g., [18,19,25,26,33,55]) solve a pair of boundary integral equations that directly treat the Laplacian of the solution and the normal derivative of the Laplacian on the boundary as two unknowns. With the indirect methods (e.g., [5,44,45]), which often derive equations by the Chakrabarty and Almansi formula, unknowns of the reformulated boundary integral equations are intermediate variables and do not have physical meaning. Most BIEMs solve a system of two boundary integral equations while the indirect methods in [46–48] work with scalar boundary integral equations. As for other PDEs [8,56], BIEMs typically yield dense matrices and become computationally efficient only when they are accelerated by fast summation techniques such as the fast multipole method [35,41], the fast Fourier–Galerkin method [50] or the tree algorithm based on randomized factorization for low rank matrices [49]. For the biharmonic equation with a non-trivial non-homogeneous source, BIEMs involve expensive volume integrals, losing the advantage of dimension reduction [12]. Besides, due to the singularity of the fundamental solutions of the harmonic and biharmonic operators, BIEMs often need to carefully handle singular, nearly singular or hyper-singular boundary integrals. Otherwise, accuracy or stability may become an issue.

In addition to the three main classes of numerical methods above for the biharmonic equation, alternative methods such as those based on complex variable theory [13,63], spectral or pseudo-spectral methods [6,32,38,42] and methods of fundamental solutions [51–54,57] exist in the literature. In particular, the methods of fundamental solutions have gained some popularity over the years [15,31,61,67,73]. Strengths and limits of methods of fundamental solutions are quite similar to BIEMs except the issue of integral singularity associated with BIEMs is replaced by the ill-conditioning property of discrete equations in methods of fundamental solutions.

This work will solve the biharmonic equation on irregular domains with a kernel-free boundary integral (KFBI) method. The KFBI method is a Cartesian grid based boundary integral method, which is originally proposed for second-order elliptic PDEs [83]. It solves boundary value or interface problems in the framework of BIMs but does not need to know or compute the integral kernel, the fundamental solution or Green's function of the elliptic operator. The method evaluates a boundary or volume integral by solving an equivalent simple interface problem with fast elliptic solvers on Cartesian grids [84,85]. It does not have singularity issues associated with the standard (traditional) BIMs. Recent version of the KFBI method for the modified Helmholtz equation [81,82] has fourth-order accuracy.

This study is further development of the KFBI method for the biharmonic equation. As in Jeon's paper [46], this work also decomposes the biharmonic equation into two Poisson equations, assumes the solution to one Poisson equation, which has no boundary conditions, as the sum of a volume integral with a double layer boundary integral, and applies Green's third identity to derive a scalar boundary integral equation from the other Poisson equation that are subject to two boundary conditions. The difference from Jeon's approach is that Green's functions in this work are defined on a rectangle. Their analytical expressions are hard to obtain or compute. As the KFBI method adapts the Cartesian grid based indirect evaluation technique, singularity issues associated with other BIEMs are removed. The present approach evaluates a boundary or volume integral by solving a corresponding equivalent interface problem by the standard five-point or the nine-point compact finite difference scheme on Cartesian grids. The resulting discrete equations are calculated by fast Fourier transform (FFT) based elliptic solvers.

The proposed KFBI method avoids generation of unstructured grids as required by standard FEMs and does not suffer from singularity issues associated with traditional BIEMs. The resulting discrete equations does not have ill-conditioning property. This work solves the biharmonic equation in two different types of computational domains. The first one is a bounded irregular domain and the second one is a singly periodic irregular domain. The latter case has not been considered yet with BIEM in the literature.

The remainder of this paper is organized as follows. In Sect. 2, the biharmonic Dirichlet problem on an irregular bounded domain and the biharmonic singly periodic problem in a periodic curved pipe are described. Section 3 is devoted to the KFBI method, including the reformulated BIE and implementation details. Numerical examples of the biharmonic prob-

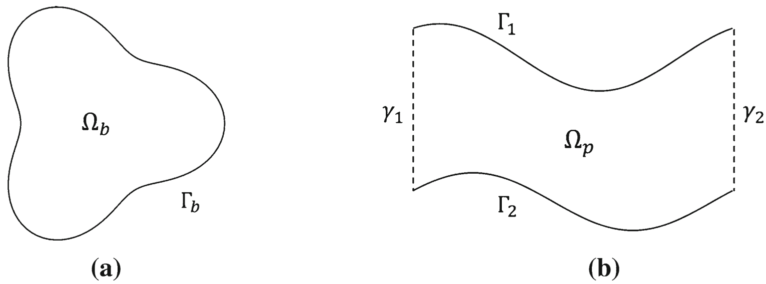


Fig. 1 Problem domains: **a** Ω_b is an irregular bounded domain; **b** Ω_p is a periodic curved pipe with the periodic boundary condition imposed on the fictitious cutting surfaces γ_1 and γ_2

lems with both second-order and fourth-order KFBI methods are given in Sect. 4. Finally, Sect. 5 summarizes the proposed method and discusses its potential improvement and extensions for the future work.

2 Boundary Value Problems

We solve the biharmonic equation on planar irregular domains and consider two cases: the first one is on a bounded irregular domain and another one is on a singly periodic irregular domain, modeling a periodic curved pipe.

Let $\Omega_b \subset \mathbb{R}^2$ be a bounded irregular domain with smooth boundary $\Gamma_b = \partial\Omega_b$ on the plane (see Fig. 1a for illustration). The first biharmonic boundary value problem reads

$$\begin{aligned} \triangle^2 u &= f_b \quad \text{in } \Omega_b, \\ u &= g_b \quad \text{on } \Gamma_b, \\ u_{\mathbf{n}} &= j_b \quad \text{on } \Gamma_b. \end{aligned} \tag{1}$$

Here, u is the unknown function and f_b, g_b and j_b are sufficiently smooth known functions; $\mathbf{n}$ is the unit outward normal on Γ_b; $u_{\mathbf{n}} = \partial u/\partial\mathbf{n}$ is the normal derivative of u.

Let $\Omega_p \subset \mathbb{R}^2$ be a singly periodic irregular domain (see Fig. 1b for illustration) and $\Gamma_p = \Gamma_1 \cup \Gamma_2$ with Γ_1 and Γ_2 denoting the upper and lower parts of the boundary, both of which are assumed to be smooth. Let γ_1 and γ_2 be the fictitious cutting surfaces of the singly periodic domain Ω_p. The second biharmonic boundary value problem for the unknown function $u = u(\mathbf{x})$ with $\mathbf{x} = (x_1, x_2)^T$ reads

$$\begin{aligned} \triangle^2 u &= f_p \quad \text{in } \Omega_p, \\ u &= g_p \quad \text{on } \Gamma_p, \\ u_{\mathbf{n}} &= j_p \quad \text{on } \Gamma_p, \\ u(\mathbf{x}) &= u(\mathbf{x}+\mathbf{L}) \quad \text{on } \bar{\Omega}_p. \end{aligned} \tag{2}$$

Here, f_p, g_p and j_p are also sufficiently smooth known functions; $\mathbf{n}$ is the unit outward normal on Γ_p; $u_{\mathbf{n}} = \partial u/\partial\mathbf{n}$ denotes the normal derivative of u; $\mathbf{L} = (L, 0)^T$, whose first component L denotes a period along the horizontal direction of the domain and second component vanishes, implying the single periodicity of u; $\bar{\Omega}_p = \Omega_p \cup \Gamma_p$ is the closure of Ω_p.

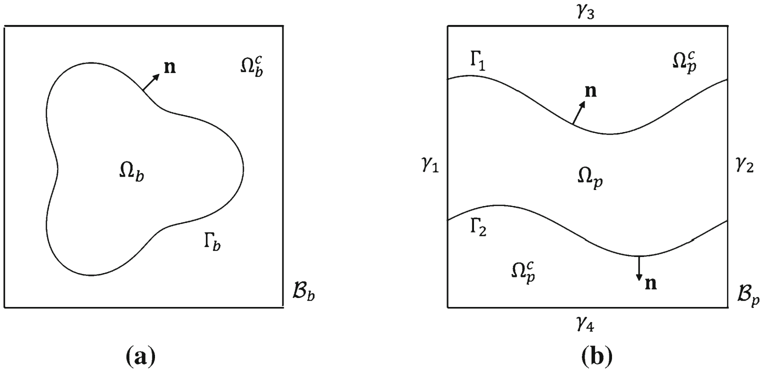


Fig. 2 Computational domains Ω_b and Ω_p are embedded in lager rectangles $\mathcal{B}_b$ and $\mathcal{B}_p$, respectively

3 Kernel-Free Boundary Integral Method

Unlike the traditional boundary integral method [45–47] for the biharmonic equation, the kernel-free boundary integral method does not work with the fundamental solution of the biharmonic equation in the free space. Instead, it works with Green's functions in a bounded domain. In this work, the biharmonic boundary value problems are first reformulated as boundary integral equations (BIEs) in terms of Green's functions associated with the Laplace operator in a rectangle and then the BIEs are solved with a Cartesian grid method.

3.1 Green's Functions

Let $\mathcal{B}_b$ and $\mathcal{B}_p$ be two rectangles, which respectively embed the simple bounded irregular domain Ω_b and the singly periodic irregular domain Ω_p so that Γ_b and Γ_p become interfaces in the rectangles. In the latter case, the fictitious boundaries γ_1 and γ_2 are part of the right and left edges of $\mathcal{B}_p$. See Fig. 2 for illustration.

Let $G_b(\mathbf{x}, \mathbf{y})$ with $\mathbf{y} = (y_1, y_2)^T \in \mathbb{R}^2$ be Green's function associated with the Laplace operator in $\mathcal{B}_b$ that satisfies

$$\begin{aligned} \triangle_{\mathbf{y}} G_b(\mathbf{x}, \mathbf{y}) &= \delta(\mathbf{x} - \mathbf{y}), \qquad \mathbf{y} \in \mathcal{B}_b, \\ G_b(\mathbf{x}, \mathbf{y}) &= 0, \mathbf{y} \in \partial \mathcal{B}_b, \end{aligned} \tag{3}$$

for each $\mathbf{x} \in \mathcal{B}_b$. Let $G_p(\mathbf{x}, \mathbf{y})$ be Green's function associated with the Laplace operator in $\mathcal{B}_p$ that satisfies

$$\begin{aligned} \triangle_{\mathbf{y}} G_p(\mathbf{x}, \mathbf{y}) &= \delta(\mathbf{x} - \mathbf{y}), && \mathbf{y} \in \mathcal{B}_p, \\ G_p(\mathbf{x}, \mathbf{y}) &= G_p(\mathbf{x}, \mathbf{y} + \mathbf{L}), && \mathbf{y} \in \bar{\mathcal{B}}_p, \\ G_p(\mathbf{x}, \mathbf{y}) &= 0, && \mathbf{y} \in \gamma_3 \cup \gamma_4, \end{aligned} \tag{4}$$

for each $\mathbf{x} \in \mathcal{B}_p$. Here, $\bar{\mathcal{B}}_p = \mathcal{B}_p \cup \partial \mathcal{B}_p$ is the closure of $\mathcal{B}_p$.

3.2 Boundary Integral Equation Formulation

As the boundary integral equations to be reformulated from the biharmonic BVPs (1) and (2) above have the same form, for conciseness, from now on we will omit the subscripts "b" and "p" in the symbols for the domains and functions and unify them as Ω, Γ, $\mathcal{B}$, f, g, j and $G(\mathbf{x}, \mathbf{y})$. That is, let Ω, Γ and $\mathcal{B}$ represent the problem domain, domain boundary and the larger rectangle, respectively; f, g, and j be the given functions in the corresponding biharmonic BVP; $G(\mathbf{x}, \mathbf{y})$ be Green's function in $\mathcal{B}$.

We work with the splitting technique to decouple the biharmonic equation into two Poisson equations. Let

$$v = \triangle u \qquad \text{in} \quad \Omega.$$

Then each of the biharmonic BVPs (1) and (2) is divided into two subproblems:

$$\begin{cases} \triangle u = v & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \Gamma, \\ u_{\mathbf{n}} = j & \text{on} \quad \Gamma, \end{cases} \tag{5}$$

and

$$\triangle v = f \qquad \text{in} \quad \Omega. \tag{6}$$

The first subproblem (5) has two boundary conditions and is over-determined while the second subproblem (6) has no boundary condition and is under-determined. They are coupled by the intermediate variable v. Fortunately, Green's third identity holds,

$$\int_{\Omega} G(\mathbf{x}, \mathbf{y}) \triangle u(\mathbf{y})\, d\mathbf{y} + \int_{\Gamma} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} u(\mathbf{y})\, ds_{\mathbf{y}} - \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}}\, ds_{\mathbf{y}}$$
$$= \begin{cases} u(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \frac{1}{2}\, u(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases}$$

Here, $\mathbf{n}_{\mathbf{y}}$ denotes the unit outward normal vector at point $\mathbf{y} \in \Gamma$. Plugging the PDE, the Dirichlet and Neumann boundary data $u = g$ and $\partial_{\mathbf{n}} u = j$ in the first subproblem (5) into Green's third identity above, one gets

$$\begin{aligned} &\int_{\Omega} G(\mathbf{x}, \mathbf{y}) v(\mathbf{y})\, d\mathbf{y} + \int_{\Gamma} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} g(\mathbf{y})\, ds_{\mathbf{y}} - \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) j(\mathbf{y})\, ds_{\mathbf{y}} \\ &\quad = \begin{cases} u(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \frac{1}{2}\, g(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases} \end{aligned} \tag{7}$$

As v satisfies the Poisson equation in the second subproblem (6), it can be regarded as the summation of a Newton potential with respect to the known function f and a double layer potential with an unknown density ξ, which is defined on the domain boundary Γ and to be determined later, i.e.,

$$v(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} + \int_{\Gamma} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} \xi(\mathbf{y})\, ds_{\mathbf{y}}. \tag{8}$$

Substituting (8) into (7) yields

$$\begin{aligned} &\int_{\Omega} G(\mathbf{x}, \mathbf{y}) \int_{\Omega} G(\mathbf{y}, \mathbf{z}) f(\mathbf{z})\, d\mathbf{z}\, d\mathbf{y} + \int_{\Omega} G(\mathbf{x}, \mathbf{y}) \int_{\Gamma} \frac{\partial G(\mathbf{y}, \mathbf{z})}{\partial \mathbf{n}_{\mathbf{z}}} \xi(\mathbf{z})\, ds_{\mathbf{z}}\, d\mathbf{y} \\ &\quad + \int_{\Gamma} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} g(\mathbf{y})\, ds_{\mathbf{y}} - \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) j(\mathbf{y})\, ds_{\mathbf{y}} = \begin{cases} u(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \frac{1}{2}\, g(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases} \end{aligned} \tag{9}$$

The identity (9) gives the representation formula for the solution u as well as the boundary integral equation for ξ to be solved. For simplicity, following the notations for the volume and boundary integrals in [82,85], one may rewrite (9) as

$$u = \mathcal{M}g - \mathcal{L}j + \mathcal{G}(\mathcal{G}f) + \mathcal{G}(\mathcal{M}\xi) \quad \text{in } \Omega, \tag{10}$$

$$\mathcal{G}(\mathcal{M}\xi) = \left(\frac{1}{2} - \mathcal{M}\right)g + \mathcal{L}j - \mathcal{G}(\mathcal{G}f) \quad \text{on } \Gamma. \tag{11}$$

Here, $\mathcal{M}$, $\mathcal{L}$, and $\mathcal{G}$ are respectively the double layer, single layer boundary and volume integrals, given by

$$\begin{aligned} \mathcal{M}\varphi(\mathbf{x}) &= \int_{\Gamma} \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n_y}} \varphi(\mathbf{y})\, ds_{\mathbf{y}}, \\ \mathcal{L}\psi(\mathbf{x}) &= \int_{\Gamma} G(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\, ds_{\mathbf{y}}, \\ \mathcal{G}f(\mathbf{x}) &= \int_{\Omega} G(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d\mathbf{y}, \end{aligned} \tag{12}$$

for functions φ, ψ and f. Note that the operator notations $\mathcal{M}$, $\mathcal{L}$, and $\mathcal{G}$ would also be endowed with the subscripts "b'' or "p'' in the ensuing subsection, together with the symbols Ω and $G(\mathbf{x},\mathbf{y})$ in the definition (12), when the specific biharmonic BVP (1) or (2) is under consideration. With the jump relations of the double layer potential, single layer potential and Newton potential [40], the BIE (11) can be reinterpreted in a clearer form

$$(\mathcal{G}(\mathcal{M}\xi))^{+} = g - (\mathcal{M}g)^{+} + (\mathcal{L}j)^{+} - (\mathcal{G}(\mathcal{G}f))^{+} \quad \text{on } \Gamma. \tag{13}$$

Or alternatively,

$$(\mathcal{G}(\mathcal{M}\xi))^{-} = -(\mathcal{M}g)^{-} + (\mathcal{L}j)^{-} - (\mathcal{G}(\mathcal{G}f))^{-} \quad \text{on } \Gamma. \tag{14}$$

Here, superscript "+" or "−" suggests that the value of the function on the interface is one-sided limit from the domain Ω or Ω^c, respectively.

The BIE (13) or (14) can be solved by the generalized minimal residual (GMRES) method [72] after the composite integral operator $\mathcal{G}\mathcal{M}$ is further discretized. Once a numerical solution to the BIE is computed, it is put into the representation formula (10) to obtain an approximation for u. In other words, the solution procedure of this work primarily consists of two components: (1) evaluation of boundary and volume integrals in Ω; (2) interpolation for boundary values of the integrals on Γ. These two parts, which are essential in approximation for each term of (10) and (13) (or (14) as an alternative), will be respectively discussed in the following two subsections.

Before proceeding, we give two remarks about the biharmonic singly periodic problem (2) in Ω_p. First, any involved function defined on the domain boundary $\Gamma_p = \Gamma_1 \cup \Gamma_2$ is actually piece-wisely defined. In the BIE (13) [or (14)], which takes a general form, the functions g and j are the known Dirichlet and Neumann boundary data provided $g|_{\Gamma_1} = g_1$, $g|_{\Gamma_2} = g_2$, $j|_{\Gamma_1} = j_1$, $j|_{\Gamma_2} = j_2$, where g_1, g_2, j_1, j_2 are given in Sect. 2. Second, any function defined on the domain boundary Γ_p, say ϕ, despite piece-wisely defined, should be regarded as a whole. From now on, we use ϕ_1 and ϕ_2 to denote the restriction of ϕ on Γ_1 and Γ_2 respectively, i.e., $\phi|_{\Gamma_1} = \phi_1$, $\phi|_{\Gamma_2} = \phi_2$.

3.3 Evaluation for Boundary and Volume Integrals

In this subsection, we present the Cartesian grid based indirect method for evaluating boundary and volume integrals. In the method, integrals are regarded as solutions to some equivalent simple interface problems. Analytical expressions of Green's functions are no longer needed. This is a major difference of the KFBI method from the traditional BIEMs.

3.3.1 Equivalent Interface Problems

For the biharmonic Dirichlet problem (1) defined in irregular bounded domain Ω_b with the corresponding Green's function $G_b(\mathbf{x}, \mathbf{y})$, equivalent interface problem of the boundary integral $\mathcal{M}_b\varphi$, $-\mathcal{L}_b\psi$ and the volume integral $\mathcal{G}_b f$ can be interpreted in an unified form

$$\begin{aligned} \triangle v &= F && \text{in } \Omega_b \cup \Omega_b^c, \\ [v] &= \Phi && \text{on } \Gamma_b, \\ [v_{\mathbf{n}}] &= \Psi && \text{on } \Gamma_b, \\ v &= 0 && \text{on } \partial\mathcal{B}_b. \end{aligned} \tag{15}$$

Here, F, Φ and Ψ for potentials of different type have different values [76,81,85]. Specifically, for double layer potential $\mathcal{M}_b\varphi$, $F = 0$, $\Phi = \varphi$, $\Psi = 0$. For single layer potential $\mathcal{L}_b\psi$, $F = 0$, $\Phi = 0$, $\Psi = \psi$. While for Newton potential $\mathcal{G}_b f$, F equals f inside Ω_b and vanishes in Ω_b^c, $\Phi = 0$, $\Psi = 0$.

Similarly, for the biharmonic singly periodic problem (2), the formally unified equivalent interface problem of the integrals $\mathcal{M}_p\varphi$, $-\mathcal{L}_p\psi$ and $\mathcal{G}_p f$ reads

$$\begin{aligned} \triangle v &= F && \text{in } \Omega_p \cup \Omega_p^c, \\ [v] &= \Phi_i && \text{on } \Gamma_i, i = 1, 2, \\ [v_{\mathbf{n}}] &= \Psi_i && \text{on } \Gamma_i, i = 1, 2, \\ v &= 0 && \text{on } \partial\mathcal{B}_p, \\ v(\mathbf{x}) &= v(\mathbf{x} + \mathbf{L}) && \text{for } \mathbf{x} \in \bar{\mathcal{B}}_p. \end{aligned} \tag{16}$$

Here, for double layer potential $\mathcal{M}_p\varphi$, $F = 0$, $\Phi_i = \varphi|_{\Gamma_i} = \varphi_i$, $\Psi_i = 0$, $i = 1, 2$. For single layer potential $\mathcal{L}_p\psi$, $F = 0$, $\Phi_i = 0$, $\Psi_i = \psi|_{\Gamma_i} = \psi_i$, $i = 1, 2$. While for Newton potential $\mathcal{G}_p f$, F equals to f inside Ω_p and vanishes in Ω_p^c, $\Phi_i = 0$, $\Psi_i = 0$, $i = 1, 2$.

Therefore, to evaluate the volume or boundary integral in (11) for the biharmonic problem (1) or (2), we turn to solving the interface problem (15) or (16) with the source term F and the jump Φ, Ψ or Φ_i, Ψ_i $(i = 1, 2)$ on the interface determined by the corresponding values given above.

3.3.2 Cartesian Grid Method for Interface Problem

The interface problems (15) and (16) are solved by a Cartesian grid method, whose second-order version for variable coefficients elliptic PDEs on irregular bounded domain is proposed by Ying and Wang [85]. A fourth-order version of the method for the modified Helmholtz equation is proposed in [81]. It is straightforward to apply the methods [81,85] to get second-order and fourth-order accurate solutions to the interface problem (15). For the interface problem (16), special manipulation is needed in dealing with the disconnected interfaces Γ_1 and Γ_2 as well as the periodic boundary conditions.

To begin with, the computational domain Ω_p is partitioned into Cartesian grid with the horizontal and vertical mesh parameters uniformed as h. Then the interfaces Γ_1 and Γ_2 are discretized by a set of quasi-uniformly spaced points, respectively. Next, we use the standard five-point finite difference scheme or the nine-point compact difference scheme [71,74] to discretize the control equation $\triangle v = F$. The established linear system needs to be corrected as the local truncation error near the interface Γ_1 and Γ_2 is large. Here, correction technique is completely the same as that needed in solving the interface problem (15) except the two interfaces Γ_1 and Γ_2 are treated separately. The correction term with respect to the grid points near the interface Γ_i, $(i = 1, 2)$ contains the jumps of partial derivatives of v which can be derived according to the reiterated conditions as below

$$\begin{aligned} \triangle v &= F \qquad \text{in } \Omega_p \cup \Omega_p^c, \\ [v] &= \Phi_i \qquad \text{on } \Gamma_i, \\ [v_{\mathbf{n}}] &= \Psi_i \qquad \text{on } \Gamma_i. \end{aligned} \tag{17}$$

Refer to [81,82,85] for implementation details for the second-order or fourth-order versions of the method. The resulting modified system is finally solved by an FFT solver with the horizontal direction subject to the periodic boundary condition. In this way, an approximation of v, and more importantly, its equivalent boundary or volume integral is obtained at the Cartesian grid nodes.

3.4 Interpolation for Boundary Values

The purpose of solving the interface problems above is to indirectly evaluate a boundary or volume integral at discretization points on the interfaces or the boundary of the original problem domain. After the solution to an interface problem is calculated at Cartesian grid nodes, one needs to interpolate them to get its values at points on the interfaces. Once again, the interpolation methods proposed in [81,85] can be directly applied for the biharmonic Dirichlet problem (1) in the bounded irregular domain. While for the biharmonic problem on the singly periodic irregular domain (2), interpolation for boundary values at Γ_1 and Γ_2 should be performed separately, since the required jumps of partial derivative on different boundaries are derived according to different interface conditions. However, the details are omitted to limit the number of pages.

4 Numerical Results

In this section, we present numerical results for both the biharmonic Dirichlet problem on the bounded irregular domain and the problem on the singly periodic irregular domain with the proposed second-order and fourth-order kernel-free boundary integral methods, respectively.

For numerical tests of the biharmonic Dirichlet problem, we use a rotated ellipse and a rotated star-shaped domain as the computational domains. The rotated ellipse is given by

$$\begin{cases} x = c_x + r_a \cos\theta \cos\alpha - r_b \sin\theta \sin\alpha \\ y = c_y + r_a \cos\theta \sin\alpha + r_b \sin\theta \cos\alpha \end{cases} \qquad \text{for } \theta \in [0, 2\pi),$$

where (c_x, c_y) is the coordinate of the rotated ellipse center, α is the rotation angle, r_a and r_b are the major and minor radius of the ellipse, respectively. The bounding box $\mathcal{B}$ for the interface problems is set to be $\mathcal{B} = (-\pi + c_x, \pi + c_x) \times (-\pi + c_y, \pi + c_y)$.

The star-shaped domain is given by

$$\begin{cases} x = c_x + p(\theta) \cos\alpha - q(\theta) \sin\alpha \\ y = c_y + p(\theta) \sin\alpha + q(\theta) \cos\alpha \end{cases} \qquad \text{for } \theta \in [0, 2\pi),$$

with

$$\begin{cases} p(\theta) = \big[(1-c) + c \cdot \cos(m\theta)\big] \cos\theta \\ q(\theta) = \big[(1-c) + c \cdot \cos(m\theta)\big] \sin\theta \end{cases},$$

where m is the fold number and $c = 0.2$ remains unchanged. The corresponding bounding box $\mathcal{B}$ is chosen to be $(-1.2 + c_x, 1.2 + c_x) \times (-1.2 + c_y, 1.2 + c_y)$.

For numerical tests of the biharmonic singly periodic problem, we set both the upper and lower boundary of the curved pipe as the sine curve in the following form,

$$\begin{cases} x = -1.0 + \dfrac{\theta}{\pi} \\ y = A \cdot \sin(\kappa\,\theta + \alpha) + D \end{cases} \quad \text{for} \quad \theta \in [0,\ 2\pi),$$

where A is the amplitude, κ is the wave number, α is the phase angle, and D is the mean height of the sine curve. The bounding box $\mathcal{B}$ for the interface problems is fixed to be $\mathcal{B} = (-1, 1) \times (-1, 1)$.

In all numerical examples, the curves are discretized into evenly spaced points with respect to the parameter θ and the discrete boundary integral equations are solved with the generalized minimal residual (GMRES) [72] method. In each test, the GMRES iteration starts with the trivial zero initial guess. For different numerical examples with respect to different types of biharmonic problems and under different accuracy requirements, the tolerance ϵ for terminating the GMRES iteration is specified exclusively so that the accuracy order can be observed on the constantly refined Cartesian grids. Numerical results are illustrated in tables and figures. Each table has five rows, presenting the grid sizes used in the Cartesian grid, the total number of discretization points on the boundary curve, the numbers of GMRES iteration in solving the BIE, the discrete maximum error of the numerical solution at the interior grid nodes and the CPU times (in seconds) by a MacBook Pro laptop computer, which has a 2.5 GHz Intel Core i7 processor. Each figure corresponds to an example, showing 36 isolines of the numerical solution in the interior and exterior of the problem domain, respectively.

4.1 Biharmonic Dirichlet Problem

Example 1 This example solves the non-homogeneous biharmonic Dirichlet problem on the rotated ellipse with the KFBI method. The tolerance in the GMRES iteration for the second-order and fourth-order KFBI methods is set to be 10^{-4} and 10^{-8}, respectively. Parameters of the rotated ellipse are as follows:

$$c_x = 0.2,\ \ c_y = 0.4,\ \ r_a = 0.8\pi,\ \ r_b = 0.45\pi,\ \ \alpha = 6\pi/7.$$

The biharmonic Dirichlet and Neumann boundary conditions on the rotated ellipse are chosen so that the exact solution reads

$$u(x, y) = x^2(1 - x^2)y^2(1 - y^2).$$

Numerical results with second-order and fourth-order accuracy are summarized in Tables 1 and 2, respectively (Figs. 3, 4, 5, 6).

Example 2 This example solves the homogeneous biharmonic Dirichlet problem on a rotated star-shaped domain with the KFBI method. The tolerance in the GMRES iteration for the second-order and fourth-order KFBI methods is set to be 10^{-8} and 10^{-9}, respectively. Parameters of the rotated star-shaped domain are as follows:

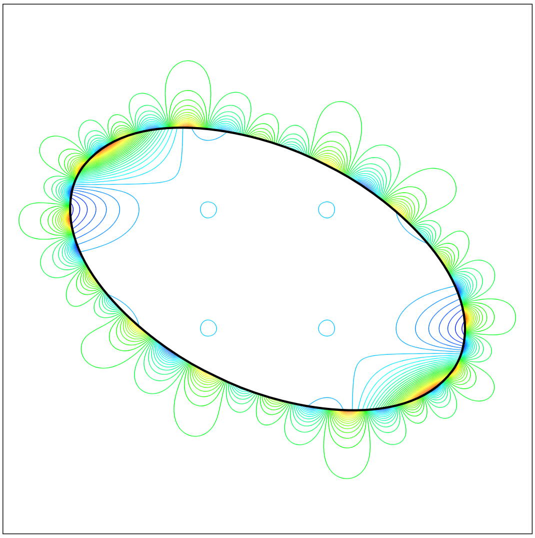
Table 1 Numerical results with second-order accuracy of Example 1

Grid size	64 × 64	128 × 128	256 × 256	512 × 512	1024 × 1024
M	64	128	256	512	1024
#GMRES	53	87	46	13	13
CPU (s)	7.76E−1	3.42E+0	9.50E+0	3.01E+1	1.25E+2
$\|\mathbf{e}_h\|_\infty$	7.07E−2	2.47E−2	2.13E−3	5.53E−4	1.61E−4
Order	–	1.52	3.54	1.94	1.78

Table 2 Numerical results with fourth-order accuracy of Example 1

Grid size	64 × 64	128 × 128	256 × 256	512 × 512	1024 × 1024
M	64	128	256	512	1024
#GMRES	31	41	47	33	26
CPU (s)	9.61E−1	2.37E+0	8.10E+0	2.13E+1	6.82E+1
$\|\mathbf{e}_h\|_\infty$	2.65E−3	1.28E−4	4.19E−6	1.85E−7	1.47E−8
Order	–	4.37	4.93	4.50	3.65

Fig. 3 Isolines of the numerical solution with fourth-order accuracy in Example 1 on the 1024 × 1024 grid



$$m = 5,\ \ c_x = 2.0,\ \ c_y = 2.0,\ \ \alpha = 0.75\pi.$$

The biharmonic Dirichlet and Neumann boundary conditions are chosen so that the exact solution reads

$$u(x, y) = \left[(x + 1.2)^2 + (y + 1.6)^2\right]\left[\log\sqrt{(x + 1.2)^2 + (y + 1.6)^2} - 1.0\right].$$

Numerical results with second-order and fourth-order accuracy are summarized in Tables 3 and 4, respectively.

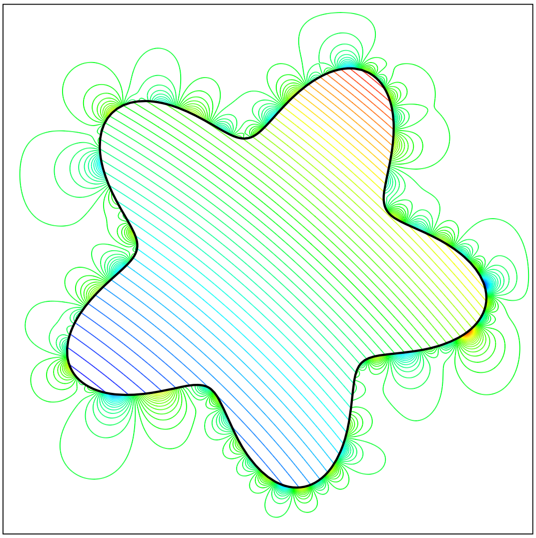


Fig. 4 Isolines of the numerical solution with fourth-order accuracy in Example 2 on the 1024×1024 grid

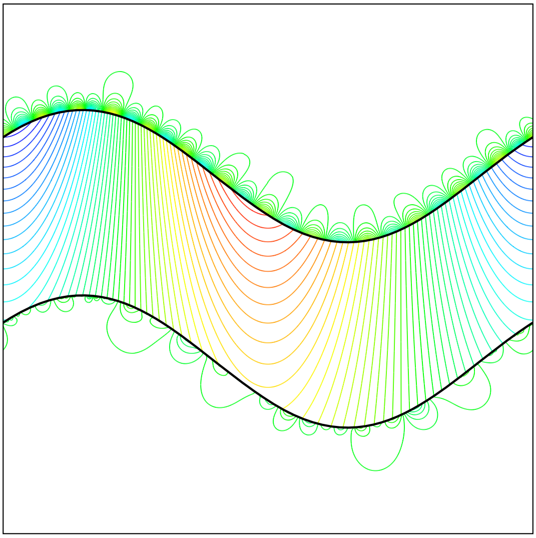


Fig. 5 Isolines of the numerical solution with fourth-order accuracy in Example 3 on the 1024×1024 grid

4.2 Biharmonic Singly Periodic Problem

Example 3 This example solves the non-homogeneous biharmonic singly periodic problem on the periodic curved pipe with the KFBI method. The tolerance in the GMRES iteration for the second-order and fourth-order KFBI methods is set to be 10^{-7} and 10^{-11}, respectively. Parameters of the upper boundary (l_1) and lower boundary (l_2) are as follows:

$$\begin{cases} l_1 : A_1 = 0.25, \ \kappa_1 = 1, \ \alpha_1 = 0.2\pi, \ D_1 = 0.35 \\ l_2 : A_2 = 0.25, \ \kappa_2 = 1, \ \alpha_2 = 0.2\pi, \ D_2 = -0.35 \end{cases}$$

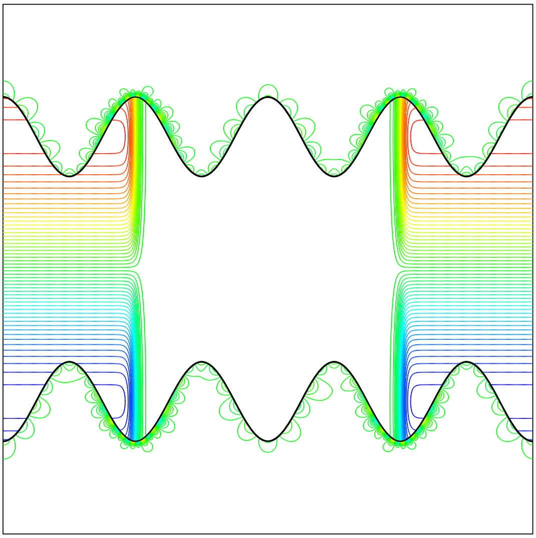


Fig. 6 Isolines of the numerical solution with second-order accuracy in Example 4 on the 1024×1024 grid

Table 3 Numerical results with second-order accuracy of Example 2

Grid size	64×64	128×128	256×256	512×512	1024×1024
M	64	128	256	512	1024
#GMRES	70	87	181	72	25
CPU (s)	7.02E−1	2.65E+0	1.92E+1	4.31E+1	1.13E+2
$\\|\mathbf{e}_h\\|_\infty$	2.02E−3	1.83E−4	1.75E−5	5.03E−7	5.29E−8
Order	–	3.46	3.38	5.12	3.25

Table 4 Numerical results with fourth-order accuracy of Example 2

Grid size	64×64	128×128	256×256	512×512	1024×1024
M	64	128	256	512	1024
#GMRES	36	49	59	53	30
CPU (s)	1.02E+0	2.53E+0	8.78E+0	3.17E+1	7.07E+1
$\\|\mathbf{e}_h\\|_\infty$	3.74E−3	1.24E−4	7.84E−6	2.28E−7	2.71E−9
Order	–	4.91	3.98	5.10	6.39

The biharmonic Dirichlet and Neumann boundary conditions on periodic curved pipe are chosen so that the exact solution reads

$$u(x, y) = \cos(\pi x)e^y.$$

Numerical results with second-order and fourth-order accuracy are summarized in Tables 5 and 6, respectively.

Example 4 This example solves the non-homogeneous biharmonic singly periodic problem on the periodic curved pipe with the KFBI method. The tolerance in the GMRES iteration for

Table 5 Numerical results with second-order accuracy of Example 3

Grid size	64 × 64	128 × 128	256 × 256	512 × 512	1024 × 1024
M	64	128	256	512	1024
#GMRES	28	33	30	16	16
CPU (s)	1.19E+0	5.43E+0	2.23E+1	4.81E+1	2.18E+2
$\|\mathbf{e}_h\|_\infty$	1.61E−4	3.71E−5	9.46E−6	2.57E−6	5.91E−7
Order	–	2.12	1.97	1.88	2.12

Table 6 Numerical results with fourth-order accuracy of Example 3

Grid size	64 × 64	128 × 128	256 × 256	512 × 512	1024 × 1024
M	64	128	256	512	1024
#GMRES	32	40	45	40	28
CPU (s)	3.28E+0	8.33E+0	3.67E+1	1.38E+2	4.57E+2
$\|\mathbf{e}_h\|_\infty$	5.20E−5	3.92E−7	2.26E−8	6.97E−10	3.00E−11
Order	–	7.05	4.12	5.02	4.54

Table 7 Numerical results with second-order accuracy of Example 4

Grid size	64 × 64	128 × 128	256 × 256	512 × 512	1024 × 1024
M	64	128	256	512	1024
#GMRES	26	27	32	30	28
CPU (s)	1.28E+0	3.82E+0	1.73E+1	6.54E+1	2.69E+2
$\|\mathbf{e}_h\|_\infty$	3.68E+0	1.88E−1	9.91E−3	2.55E−3	6.20E−4
Order	–	4.29	4.25	1.96	2.04

the second-order and fourth-order KFBI methods is set to be 10^{-4} and 10^{-6}, respectively. Parameters of the upper boundary (l_1) and lower boundary (l_2) are as follows:

$$\begin{cases} l_1 : A_1 = 0.15, \ \kappa_1 = 4, \ \alpha_1 = 0.5\pi, \ D_1 = 0.5 \\ l_2 : A_2 = 0.15, \ \kappa_2 = 4, \ \alpha_2 = 1.5\pi, \ D_2 = -0.5 \end{cases}$$

The biharmonic Dirichlet and Neumann boundary conditions on periodic curved pipe are chosen so that the exact solution reads

$$u(x, y) = \frac{\sin(\pi y)}{1 + e^{\frac{0.5-|x|}{\varepsilon}}}.$$

Here, ε is a small quantity. In this example, we set $\varepsilon = 0.01$. Numerical results with second-order and fourth-order accuracy are summarized in Tables 7 and 8, respectively.

5 Discussion

This work proposes second-order and fourth-order versions of a KFBI method for the biharmonic equation on both bounded irregular domains and singly periodic irregular domains.

Table 8 Numerical results with fourth-order accuracy of Example 4

Grid size	64 × 64	128 × 128	256 × 256	512 × 512	1024 × 1024
M	64	128	256	512	1024
#GMRES	26	38	58	67	63
CPU (s)	4.40E+0	1.27E+1	6.20E+1	2.41E+2	8.61E+2
$\|\mathbf{e}_h\|_\infty$	1.97E−1	2.83E−1	3.14E−3	1.37E−4	6.82E−6
Order	–	− 0.52	6.49	4.52	4.33

The method adopts the splitting approach, which decomposes the fourth order elliptic BVP into two coupled Poisson BVPs, and derives BIEs based on Green's third identity. Following the lines of the previous second-order and fourth-order accurate versions [81,85] of the KFBI method, it does not need to know or compute the fundamental solution or Green's function of the elliptic differential operator and does not directly discretize boundary or volume integrals by any numerical quadrature. Instead, it evaluates boundary and volume integrals indirectly by solving equivalent but simple interface problems on Cartesian grids and takes advantages of fast elliptic solvers after routine modification to the right hand side of the discrete system at irregular grid nodes. The KFBI method for the problems shows its strengths over traditional boundary integral equation methods. It avoids evaluating singular, nearly singular or hyper-singular integrals and is also able to efficiently solve the biharmonic equation with a non-homogeneous source.

In the derivation of BIE for the biharmonic equation, the current approach assumes the auxiliary function v, which represents $\triangle u$, as the sum of a volume integral and a double layer potential and treats the density of the double layer potential as the unknown of the scalar BIE. Alternatively, one may assume the auxiliary function v as the sum of a volume integral and a single layer potential and treats the density of the single layer potential as the unknown. Besides, the current BIE is derived based on the discontinuity property of the double layer potential. One may get a different BIE based on the continuity property of the normal derivative of the double layer potential. It is worthwhile to make further studies on performance of the different formulations based on different BIEs.

The BIEs by the KFBI method for the second-order elliptic PDEs in the previous studies [81,85] are all the Fredholm boundary integral equations of the second kind. The corresponding discrete equations are well-conditioned as long as the mesh parameter is sufficiently small. When the BIEs are solved by an iterative method, the number of iterations is essentially independent of the size of the underlying Cartesian grid and the number of unknowns. The BIEs by the method for the biharmonic equation are not second kind Fredholm equations. In this work, the discrete linear system is solved by the GMRES method. The GMRES iteration needs preconditioning to reduce the number of iterations. Although the existing iteration converges generally within an acceptable number of iterations, we expect a faster convergence rate. Common preconditioners for the GMRES iteration in references can not be directly applied since the coefficient matrix of the linear system is never formulated and actually unknown. Although matrix-vector products in the method are replaced by simple equivalent interface problems on Cartesian grids, noting that matrix vector products are approximations of boundary or volume integrals, one may seek a preconditioning technique along this direction by finding an approximate integral operator.

It is straightforward to apply the method proposed in this work to solve the steady-state Stokes equations as they can be transformed as a biharmonic equation via the vorticity stream function formulation. It is also possible to solve time-dependent Stokes equations and even the Navier–Stokes equations with the method. In particular, the biharmonic BVP in the singly periodic irregular domain may serve as a good simulation of the two dimensional incompressible channel flow. This model can be further extended to a doubly periodic situation in three space dimensions.

References

1. Adini, A., Clough, R.W.: Analysis of Plate Bending by the Finite Element Method. University of California, Berkeley (1960)
2. Arad, M., Yakhot, A., Ben-Dor, G.: A highly accurate numerical solution of a biharmonic equation. Numer. Methods Partial Differ. Equ. **13**(4), 375–391 (1997)
3. Argyris, J.H., Dunne, P.C.: The finite element method applied to fluid dynamics. In: Hewitt, B.L., Illingworth, C.R., Lock, R.C., Mangler, K.W., McDonnel, J.H., Richards, C., Walkden, F. (eds.) Computational Methods and Problems in Aeronautical Fluid Dynamics, pp. 158–197. Academic Press, London (1976)
4. Baker, G.A.: Finite element methods for elliptic equations using nonconforming elements. Math. Comput. **31**, 45–59 (1977)
5. Bhargava, R.: Solution of a biharmonic equation. Nature **201**(4918), 530 (1964)
6. Bialecki, B., Karageorghis, A.: Spectral chebyshev collocation for the Poisson and biharmonic equations. SIAM J. Sci. Comput. **32**(5), 2995–3019 (2010)
7. Bjørstad, P.: Fast numerical solution of the biharmonic Dirichlet problem on rectangles. SIAM J. Numer. Anal. **20**(1), 59–71 (1983)
8. Brebbia, C.A., Telles, J.C.F., Wrobel, L.C.: Boundary Element Techniques: Theory and Applications in Engineering. Springer, New York (2012)
9. Brenner, S., Sung, L.: C0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains. J. Sci. Comput. **22/23**(1–3), 83–118 (2005)
10. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer, Berlin (1991)
11. Buzbee, B.L., Golub, G.H., Nielson, C.W.: On direct methods for solving Poisson's equations. SIAM J. Numer. Anal. **7**(4), 627–656 (1970)
12. Camp, C.V.: Solution of the nonhomogeneous biharmonic equation by the boundary element method. Ph.D. thesis, Oklahoma State University (1987)
13. Chan, R.H., DeLillo, T.K., Horn, M.A.: The numerical solution of the biharmonic equation by conformal mapping. SIAM J. Sci. Comput. **18**(6), 1571–1582 (1997)
14. Chen, G., Li, Z., Lin, P.: A fast finite difference method for biharmonic equations on irregular domains and its application to an incompressible Stokes flow. Adv. Comput. Math. **29**(2), 113–133 (2008)
15. Chen, J.T., Wu, C.S., Lee, Y.T., Chen, K.H.: On the equivalence of the Trefftz method and method of fundamental solutions for Laplace and biharmonic equations. Comput. Math. Appl. **53**, 851–879 (2007)
16. Cheng, X.L., Han, W., Huang, H.C.: Some mixed finite element methods for biharmonic equation. J. Comput. Appl. Math. **126**, 91–109 (2000)
17. Christiansen, S.: Integral equations without a unique solution can be made useful for solving some plane harmonic problems. IMA J. Appl. Math. **16**(2), 143–159 (1975)
18. Christiansen, S.: Derivation and analytical investigation of three direct boundary integral equations for the fundamental biharmonic problem. J. Comput. Appl. Math. **91**(2), 231–247 (1998)
19. Christiansen, S., Hougaard, P.: An investigation of a pair of integral equations for the biharmonic problem. IMA J. Appl. Math. **22**(1), 15–27 (1978)
20. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
21. Ciarlet, P.G., Raviart, P.A.: A mixed finite element method for the biharmonic equation. In: Mathematical Aspects of Finite Elements in Partial Differential Equations, pp. 125–145 (1974)
22. Cockburn, B., Dong, B., Guzmán, J.: A hybridizable and superconvergent discontinuous Galerkin method for biharmonic problems. J. Sci. Comput. **40**(1–3), 141–187 (2009)
23. Collatz, L.: The Numerical Treatment of Differential Equations, vol. 60. Springer, New York (2012)
24. Conte, S.D., Dames, R.T.: An alternating direction method for solving the biharmonic equation. Math. Tables Other Aids Comput. **12**(63), 198–205 (1958)
25. Costabel, M., Lusikka, I., Saranen, J.: Comparison of three boundary element approaches for the solution of the clamped plate problem. In: Ciarlet, P.G., Lions, J.L. (eds.) Boundary Elements IX, vol. 2, pp. 19–34. Springer, Berlin (1987)

26. Costabel, M., Saranen, J.: Boundary element analysis of a direct method for the biharmonic Dirichlet problem. In: The Gohberg anniversary collection, pp. 569–587 (1989)
27. Davini, C., Pitacco, I.: An unconstrained mixed method for the biharmonic problem. SIAM J. Numer. Anal. **38**(3), 820–836 (2000)
28. Ehrlich, L.W.: Solving the biharmonic equation as coupled finite difference equations. SIAM J. Numer. Anal. **8**(2), 278–287 (1971)
29. Ehrlich, L.W., Gupta, M.M.: Some difference schemes for the biharmonic equation. SIAM J. Numer. Anal. **12**(5), 773–790 (1975)
30. Fairweather, G., Gourlay, A., Mitchell, A.: Some high accuracy difference schemes with a splitting operator for equations of parabolic and elliptic type. Numer. Math. **10**(1), 56–66 (1967)
31. Fairweather, G., Karageorghis, A.: The method of fundamental solutions for elliptic boundary value problems. Adv. Comput. Math. **9**(1–2), 69 (1998)
32. Fornberg, B.: A pseudospectral approach for polar and spherical geometries. SIAM J. Sci. Comput. **16**(5), 1071–1081 (1995)
33. Fuglede, B.: On a direct method of integral equations for solving the biharmonic Dirichlet problem. Zeitschrift für Angewandte Mathematik und Mechanik **61**(9), 449–459 (1981)
34. Glowinski, R., Pironneau, O.: Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem. SIAM Rev. **21**(2), 167–212 (1979)
35. Greenbaum, A., Greengard, L., Mayo, A.: On the numerical solution of the biharmonic equation in the plane. Physica D **60**, 216–225 (1992)
36. Greengard, L., Kropinski, M.C.: An integral equation approach to the incompressible Navier–Stokes equations in two dimensions. SIAM J. Sci. Comput. **20**(1), 318–336 (1998)
37. Hadjidimos, A.: The numerical solution of a model problem biharmonic equation by using extrapolated alternating direction implicit methods. Numer. Math. **17**(4), 301–317 (1971)
38. Heinrichs, W.: A stabilized treatment of the biharmonic operator with spectral methods. SIAM J. Sci. Stat. Comput. **12**(5), 1162–1172 (1991)
39. Hockney, R.W.: A fast direct solution of Poisson's equation using fourier analysis. JACM **12**(1), 95–113 (1965)
40. Hsiao, G.C., Wendland, W.L.: Boundary Integral Equations. Springer, New York (2008)
41. Huang, S., Liu, Y.: A fast multipole boundary element method for solving the thin plate bending problem. Eng. Anal. Bound. Elem. **37**(6), 967–976 (2013)
42. Huang, W., Tang, T.: Pseudospectral solutions for steady motion of a viscous fluid inside a circular boundary. Appl. Numer. Math. **33**(1–4), 167–173 (2000)
43. Jaswon, M., Maiti, M.: An integral equation formulation of plate bending problems. J. Eng. Math. **2**(1), 83–93 (1968)
44. Jaswon, M.A., Symm, G.T.: Integral Equation Methods in Potential Theory and Elastostatics, vol. 132. Academic Press, London (1977)
45. Jeon, Y.: An indirect boundary integral equation method for the biharmonic equation. SIAM J. Numer. Anal. **31**(2), 461–476 (1994)
46. Jeon, Y.: New boundary element formulas for the biharmonic equation. Adv. Comput. Math. **9**(1–2), 97–115 (1998)
47. Jeon, Y.: New indirect scalar boundary integral equation formulas for the biharmonic equation. J. Comput. Appl. Math. **135**(2), 313–324 (2001)
48. Jeon, Y., McLean, W.: A new boundary element method for the biharmonic equation with Dirichlet boundary conditions. Adv. Comput. Math. **19**(4), 339–354 (2003)
49. Jiang, S., Ren, R., Tsuji, P., Ying, L.: Second kind integral equations for the first kind Dirichlet problem of the biharmonic equation in three dimensions. J. Comput. Phys. **230**(19), 7488–7501 (2011)
50. Jiang, Y., Wang, B., Xu, Y.: A fast Fourier–Galerkin method solving a boundary integral equation for the biharmonic equation. SIAM J. Numer. Anal. **52**(5), 2530–2554 (2014)
51. Karageorghis, A.: Modified methods of fundamental solutions for harmonic and biharmonic problems with boundary singularities. Numer. Methods Partial Differ. Equ. **8**(1), 1–19 (1992)
52. Karageorghis, A., Fairweather, G.: The method of fundamental solutions for the numerical solution of the biharmonic equation. J. Comput. Phys. **69**(2), 434–459 (1987)
53. Karageorghis, A., Fairweather, G.: The Almansi method of fundamental solutions for solving biharmonic problems. Int. J. Numer. Methods Eng. **26**(7), 1665–1682 (1988)
54. Karageorghis, A., Fairweather, G.: The simple layer potential method of fundamental solutions for certain biharmonic problems. Int. J. Numer. Methods Fluids **9**(10), 1221–1234 (1989)
55. Katsikadelis, J., Massalas, C., Tzivanidis, G.: An integral equation of the plane problem of the theory of elasticity. Mech. Res. Commun. **4**(3), 199–208 (1977)
56. Katsikadelis, J.T.: Boundary Elements: Theory and Applications. Elsevier, Amsterdam (2002)

57. Kupradze, V.D.: A method for the approximate solution of limiting problems in mathematical physics. USSR Comput. Math. Math. Phys. **4**(6), 199–205 (1964)
58. Lai, M.C., Liu, H.C.: Fast direct solver for the biharmonic equation on a disk and its application to incompressible flows. Appl. Math. Comput. **164**(3), 679–695 (2005)
59. Lascaux, P., Lesaint, P.: Some nonconforming finite elements for the plate bending problem (Revue francaise d'automatique, informatique, recherche opérationnelle). Analyse numérique **9**(R1), 9–53 (1975)
60. Li, Z.: A fast iterative algorithm for elliptic interface problems. SIAM J. Numer. Anal. **35**(1), 230–254 (1998). Please confirm the paper title for the reference [59]
61. Li, Z.C., Lee, M.G., Chiang, J.Y., Liu, Y.P.: The Trefftz method using fundamental solutions for biharmonic equations. J. Comput. Appl. Math. **235**(15), 4350–4367 (2011)
62. Maiti, M., Chakrabarty, S.: Integral equation solutions for simply supported polygonal plates. Int. J. Eng. Sci. **12**(10), 793–806 (1974)
63. Mayo, A.: The fast solution of Poisson's and the biharmonic equations on irregular regions. SIAM J. Numer. Anal. **21**(2), 285–299 (1984)
64. Morley, L.S.D.: The triangular equilibrium element in the solution of plate bending problems. Aeronat. Q. **19**, 149–169 (1968)
65. Mu, L., Wang, J., Wang, Y., Ye, X.: A weak Galerkin mixed finite element method for biharmonic equations. In: Numerical Solution of Partial Differential Equations: Theory, Algorithms, and Their Applications, pp. 247–277. Springer, New York (2013)
66. Mu, L., Wang, J., Ye, X.: Weak Galerkin finite element methods for the biharmonic equation on polytopal meshes. Numer. Methods Partial Differ. Equ. **30**(3), 1003–1029 (2014)
67. Poullikkas, A., Karageorghis, A., Georgiou, G.: Methods of fundamental solutions for harmonic and biharmonic boundary value problems. Comput. Mech. **21**(4–5), 416–423 (1998)
68. Rim, K., Henry, A.S.: An Integral Equation Method in Plane Elasticity, vol. 779. National Aeronautics and Space Administration, Washington (1967)
69. Roache, P.J.: Computational Fluid Dynamics. Hermosa Publishers, Albuquerque (1976)
70. Roberts, J., Thomas, J.M.: Mixed and hybrid methods. In: Ciarlet, P.G., Lions, J.L. (eds.) Handbook of Numerical Analysis, vol. 2. North-Holland, Amsterdam (1991)
71. Rosser, J.B.: Nine-point difference solutions for Poisson's equation. Comput. Math. Appl. **1**(3–4), 351–360 (1975)
72. Saad, Y., Schultz, M.H.: GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems. SIAM J. Sci. Stat. Comput. **7**(3), 856–869 (1986)
73. Sakakibara, K.: Method of fundamental solutions for biharmonic equation based on Almansi-type decomposition. Appl. Math. **62**(4), 297–317 (2017)
74. Samarskii, A.A.: The Theory of Difference Schemes, vol. 240. CRC Press, Boca Raton (2001)
75. Sokolnikoff, I.S.: Mathematical Theory of Elasticity. McGraw-Hill Book Company, New York (1956)
76. Steinbach, O.: Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements. Springer, New York (2007)
77. Stephenson, J.: Single cell discretizations of order two and four for biharmonic problems. J. Comput. Phys. **55**(1), 65–80 (1984)
78. Süli, E., Mozolevski, I.: hp-Version interior penalty DGFEMs for the biharmonic equation. Comput. Methods Appl. Mech. Eng. **196**(13–16), 1851–1863 (2007)
79. Timoshenko, S., Woinowsky-Krieger, S.: Theory of Plates and Shells, 2nd edn. McGraw-Hill, New York (1987)
80. Wong, Y.S., Jiang, H.: Approximate polynomial preconditioning applied to biharmonic equations. J. Supercomput. **3**, 125–145 (1989)
81. Xie, Y., Ying, W.: A fourth-order kernel-free boundary integral method for the modified Helmholtz equation. J. Sci. Comput. (2018). https://doi.org/10.1007/s10915-018-0821-8
82. Ying, W.: A Cartesian grid-based boundary integral method for an elliptic interface problem on closely packed cells. Commun. Comput. Phys. **24**(4), 1196–1220 (2018)
83. Ying, W., Henriquez, C.S.: A kernel-free boundary integral method for elliptic boundary value problems. J. Comput. Phys. **227**(2), 1046–1074 (2007)
84. Ying, W., Wang, W.C.: A kernel-free boundary integral method for implicitly defined surfaces. J. Comput. Phys. **252**, 606–624 (2013)
85. Ying, W., Wang, W.C.: A kernel-free boundary integral method for variable coefficients elliptic PDEs. Commun. Comput. Phys. **15**(4), 1108–1140 (2014)
86. Zhang, R., Zhai, Q.: A weak Galerkin finite element scheme for the biharmonic equations by using polynomials of reduced order. J. Sci. Comput. **64**(2), 559–585 (2015)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Affiliations

Yaning Xie[1] · **Wenjun Ying**[2] · **Wei-Cheng Wang**[3]

✉ Wenjun Ying
wying@sjtu.edu.cn

Wei-Cheng Wang
wangwc@math.nthu.edu.tw

1 School of Mathematical Sciences, Shanghai Jiao Tong University, Minhang 200240, Shanghai, People's Republic of China

2 School of Mathematical Sciences, MOE-LSC and Institute of Natural Sciences, Shanghai Jiao Tong University, Minhang 200240, Shanghai, People's Republic of China

3 Department of Mathematics, National Tsing Hua University and National Center for Theoretical Sciences, Hsinchu 300, Taiwan