Journal of Computational and Applied Mathematics 327 (2018) 350–361

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The direct method of lines for elliptic problems in star-shaped domains[☆]

Zhizhang Wu [a], Zhongyi Huang [a,*], Wei-Cheng Wang [b], Yi Yang [c]

[a] Department of Mathematical Sciences, Tsinghua University, Haidian District, Beijing, China
[b] Department of Mathematics, National Tsing Hua University and National Center for Theoretical Sciences, HsinChu, 300, Taiwan
[c] Department of Electrical Engineering, Tsinghua University, Haidian District, Beijing, China

ARTICLE INFO

Article history:
Received 18 December 2016
Received in revised form 26 May 2017

Keywords:
Elliptic problems
Star-shaped domains
Methods of lines
Finite element approximation
Semi-discrete approximation

ABSTRACT

In this paper, we generalize the direct method of lines for elliptic problems in star-shaped domains. We assume that the boundary of the star-shaped domain is a closed Lipschitz curve that can be parameterized by the angular variable, so that an appropriate transformation of coordinates can be introduced. Then the elliptic problem is reduced to a variational–differential problem on a semi-infinite strip in the new coordinates. We discretize the reduced problem with respect to the angular variable and obtain a semi-discrete approximation. Then a direct method is adopted to solve the semi-discrete problem analytically. Finally, the optimal error estimate of the semi-discrete approximation is given and several numerical examples are presented to show that our method is feasible and effective for a wide range of elliptic problems.

1. Introduction

Elliptic problems with some singularities arise in many fields of scientific researches and engineering applications. For instance, the static problem of heat conduction in media with interfaces or elasticity problems in composite materials would lead to an elliptic equation with discontinuous coefficients. If corners, especially cracks, are found in the media, such problems would be modeled as elliptic problems in domains with corners/cracks. Furthermore, problems like fluid flows around obstacles may give rise to the elliptic boundary value problem in unbounded domains. Many mathematicians have devoted their efforts to these kinds of problems. Kellogg and Babuska studied elliptic problems with interfaces and singularities in [1–4]. Elliptic problems in domains with corners/cracks and unbounded domains have been studied in [5–8]. It can be seen from these results that it is not easy to obtain the numerical solutions of such problems due to the singular structures and unboundedness of the domain.

A conventional approach to elliptic problems is the finite element method, in which the domain is divided into a finite number of elements. And on each element, the solution is approximated by smooth functions, e.g., polynomials. Thus if singularities appear, it would require a large number of computations to capture the singularities of the solution since the bases are smooth. Moreover, when considering problems in unbounded domains, an artificial boundary is often introduced before applying the finite element method. Many different techniques have been developed in recent years. For elliptic problems with singularities, there are methods such as mesh refinement given in [9,10]; infinite element method applied

☆ Z. Huang was supported by the NSFC Project Nos. 11322113, 91330203. W. Wang was supported by the Ministry of Science and Technology of Taiwan (103-2115-M-007-002).

* Corresponding author.
E-mail addresses: wzz14@mails.tsinghua.edu.cn (Z. Wu), zhuang@math.tsinghua.edu.cn (Z. Huang), wangwc@math.nthu.edu.tw (W.-C. Wang), sailors2008@sina.cn (Y. Yang).

http://dx.doi.org/10.1016/j.cam.2017.06.028
0377-0427/

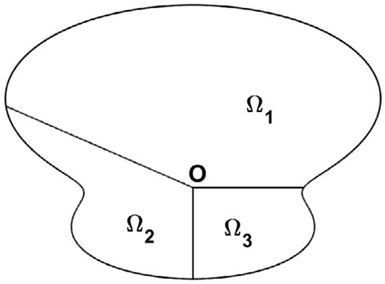

Fig. 1. Star shaped domain.

in [11,12]; the method of auxiliary mapping introduced in [13–15]; FEM with the use of singular functions discussed in [16]; the reproducing polynomial particle method presented in [17]; the multiscale finite element proposed in [18]; the discontinuous Galerkin method shown in [19]; the energy-corrected FEM described in [20]; the method based on integral equations stated in [21]; the immersed interface method illustrated in [22,23]; the immersed finite elements and the immersed finite volume element method used in [24,25], etc. To design artificial boundaries for problems in unbounded domains, several methods are adopted, e.g. absorbing boundaries and accurate radiation boundary conditions for the wave equation [26,27]; the exact boundary condition and its approximation for Laplace equation, the linear elastic system [28,29] and quasilinear elliptic equations [30]; the exact boundary condition at an artificial boundary for PDEs in an infinite cylinder [31,32] and so on. In general, the above methods solve the problem more efficiently, but there are still some limitations. For example, the method of auxiliary mapping requires a prior knowledge of the structure of the singularity at the singular point for a given problem.

Among all, the direct method of lines, which is a development of the method of lines [33,34], has been successfully applied to elliptic problems with singularities [35,36] and on unbounded domains [37,38]. The main idea of the direct method of lines is to introduce an appropriate transformation of coordinates, through which one can easily obtain a semi-discrete approximation to the original problem. Moreover, the semi-approximation can be solved by a direct method. This method requires no prior knowledge of the structure of the singularity at the singular point for a given problem. However, it has only been developed for polygonal domains.

In this paper, we want to develop the direct method of lines for elliptic problems in the 2-dimensional star-shaped domain. We assume that the boundary is a closed Lipschitz curve that can be parameterized as a piecewise smooth function of the angular variable. Then we reduce the original elliptic problem to a variational–differential problem by introducing a transformation of coordinates. The well-posedness of this variational–differential problem is discussed. And we approximate the reduced problem by discretizing it w.r.t. one variable and solve the semi-discrete problem accurately w.r.t. the other variable. An optimal error estimate can be derived for this method as well.

This paper will be arranged as follows: in Section 2, the idea of the transformation of coordinates is introduced via an elliptic interface problem. In Section 3, the numerical treatment to the reduced variational–differential problem is stated and we present the optimal error estimate of our method. Some numerical examples are given to show the feasibility and efficiency of this method in Section 4. Finally, we give a conclusion in Section 5.

2. An equivalent variational–differential formulation of an elliptic problem

In this section, we would introduce the variational–differential formulation induced by the transformation of coordinates for an elliptic interface problem in a star-shaped domain in both isotropic and anisotropic cases. The idea is similar when dealing with the problems in corner domains and unbounded domains.

2.1. Isotropic case

Let Ω be a domain with the boundary Γ in the x–y plane, where Γ is a closed Lipschitz curve which is star-shaped with respect to the origin. Without loss of generality, we assume that all interfaces meet at the origin. Suppose that Ω is divided into several subdomains $\Omega_1, \Omega_2, \dots, \Omega_J$ and that $\Omega_{k-1} \bigcap \Omega_k$ is the straight line interface $L_k = \{(r,\theta)|\theta = \theta_k, 0 \le r \le r_k\}$, where $\Omega_0 = \Omega_J$ and $\theta_1 = 0$ (see Fig. 1).

Let us consider the following interface problem:

$$\begin{aligned}
&-\nabla\cdot(P\nabla u) = 0, \quad \text{in } \Omega, (\theta \neq \theta_j),\\
&u|_\Gamma = f(\theta),\\
&u(r,\theta_j - 0) = u(r,\theta_j + 0), \quad 0 \le r \le r_j,\\
&P(\theta_j - 0)u_\theta(r,\theta_j - 0) = P(\theta_j + 0)u_\theta(r,\theta_j + 0), \quad 0 < r < r_j,
\end{aligned} \tag{2.1}$$

where $j = 1, 2, \dots, J$ and P is a scalar piecewise constant function on Ω in the isotropic case, which takes a positive constant value P_j in each Ω_j.

It is straightforward to check that problem (2.1) has a unique weak solution $u \in H_f^1(\Omega)$ for given $f \in H^{1/2}(\Gamma)$ with $H_f^1(\Omega) = \left\{ v \in H^1(\Omega) \middle| v|_\Gamma = f \right\}$, i.e., $u \in H_f^1(\Omega)$, satisfies

$$\iint_\Omega P\nabla u \cdot \nabla v dxdy = 0, \quad \forall v \in H_0^1(\Omega). \tag{2.2}$$

Suppose that Γ can be parameterized as a (piecewise) smooth function of the angular variable, $\bar{r} = \bar{r}(\phi)$, $0 \le \phi \le 2\pi$, such that $\bar{r}(0) = \bar{r}(2\pi)$ and $\bar{r}(\phi) \ge r_0 > 0$, $\forall \phi \in [0, 2\pi]$. We introduce the following transformation of coordinates on Ω,

$$(x, y) = e^\rho(\bar{r}(\phi)\cos(\phi), \bar{r}(\phi)\sin(\phi)), \quad 0 \le \phi \le 2\pi, \quad -\infty < \rho \le 0, \tag{2.3}$$

which we call the curvilinear coordinate system.

We proceed to derive the gradient and divergence operator in the curvilinear coordinates. We denote the gradient in the Cartesian coordinates by $\nabla_1 = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})^T$ and the differentiation operator in the curvilinear coordinates by $\nabla_2 = (\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \phi})^T$. Furthermore, we set $\vec{e_1}(\rho, \phi) = \partial_\rho(x(\rho, \phi), y(\rho, \phi))^T$, $\vec{e_2}(\rho, \phi) = \partial_\phi(x(\rho, \phi), y(\rho, \phi))^T$ and $E = (\vec{e_1}, \vec{e_2})^T$. Then by the chain rule, we have

$$\nabla_2 = E\nabla_1, \tag{2.4}$$

which means

$$\nabla_1 = E^{-1}\nabla_2. \tag{2.5}$$

Then for the divergence operator, we have to note that for a vector function $\vec{v} = (v_1(\rho, \phi), v_2(\rho, \phi))^T$,

$$\nabla_1 \cdot \vec{v} = \nabla_2 \cdot (E^{-T}\vec{v}) - (\nabla_2 \cdot E^{-T})\vec{v}. \tag{2.6}$$

After simple computations, we obtain

$$\nabla_1 \cdot \vec{v} = (\partial_\rho + 2, \partial_\phi + 2\bar{r}'(\phi)/\bar{r}(\phi))E^{-T}\vec{v}. \tag{2.7}$$

Thus in the curvilinear coordinates, the Laplace operator can be expressed as

$$\Delta u = \nabla_1 \cdot (\nabla_1 u) = (\partial_\rho + 2, \partial_\phi + 2\bar{r}'/\bar{r})(EE^T)^{-1}\nabla_2 u, \tag{2.8}$$

where $u = u(\rho, \phi)$ is a scalar function. Then if we introduce the following notations

$$g_{ij}(\rho, \phi) = \vec{e_i}^T \cdot \vec{e_j}, \quad i, j = 1, 2, \qquad \sqrt{g}(\rho, \phi) = (\det g(\rho, \phi))^{1/2}, \quad (g^{ij}) = (g_{ij})^{-1},$$

we can rewrite Eq. (2.8) in a more compact form

$$\Delta u = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j u), \tag{2.9}$$

where $(\partial_1, \partial_2) = (\partial_\rho, \partial_\phi)$. We further set $q^{ij}(\rho, \phi) = \sqrt{g}g^{ij}(\rho, \phi)$. It is easy to check with computations that q^{ij} is independent of ρ, i.e., $q^{ij}(\rho, \phi) = \hat{q}^{ij}(\phi)$. It follows that

$$\sqrt{g}\Delta u = \hat{q}^{11}(\phi)\partial_\rho^2 u + \partial_\rho(\hat{q}^{12}(\phi)\partial_\phi u) + \partial_\phi(\hat{q}^{12}(\phi)\partial_\rho u) + \partial_\phi(\hat{q}^{22}(\phi)\partial_\phi u), \tag{2.10}$$

with

$$\begin{pmatrix} \hat{q}^{11} & \hat{q}^{12} \\ \hat{q}^{21} & \hat{q}^{22} \end{pmatrix} = \begin{pmatrix} 1 + \left(\dfrac{\bar{r}'(\phi)}{\bar{r}(\phi)}\right)^2 & -\dfrac{\bar{r}'(\phi)}{\bar{r}(\phi)} \\ -\dfrac{\bar{r}'(\phi)}{\bar{r}(\phi)} & 1 \end{pmatrix}. \tag{2.11}$$

Furthermore, in the curvilinear coordinates, we obtain

$$\begin{aligned}
dxdy &= \sqrt{g}d\rho d\phi = e^{2\rho}\bar{r}(\phi)^2 d\rho d\phi, \\
\frac{\partial u}{\partial n}\Big|_\Gamma &= \frac{1}{\bar{r}^2\sqrt{\bar{r}'^2 + \bar{r}^2}}\left[(\bar{r}'^2 + \bar{r}^2)\frac{\partial u}{\partial \rho} - \bar{r}'\bar{r}\frac{\partial u}{\partial \phi}\right]\Big|_{\rho=0}, \\
\frac{\partial u}{\partial n}\Big|_{\phi=\theta_i^+} &= \frac{1}{e^\rho \bar{r}^2}\left[-\bar{r}'\frac{\partial u}{\partial \rho} + \bar{r}\frac{\partial u}{\partial \phi}\right]\Big|_{\phi=\theta_i^+}, \\
\frac{\partial u}{\partial n}\Big|_{\phi=\theta_i^-} &= \frac{1}{e^\rho \bar{r}^2}\left[-\bar{r}'\frac{\partial u}{\partial \rho} + \bar{r}\frac{\partial u}{\partial \phi}\right]\Big|_{\phi=\theta_i^-}.
\end{aligned}$$

Since P is a piecewise constant function, the interface problem (2.1) is reduced to the following boundary value problem with discontinuous coefficients on the domain $\tilde{\Omega} = \{(\rho, \phi)| -\infty < \rho \leq 0, 0 \leq \phi \leq 2\pi\}$ in the curvilinear coordinates:

$$\begin{aligned}
&\hat{q}^{11}(\phi)\partial_\rho^2 u + \partial_\rho(\hat{q}^{12}(\phi)\partial_\phi u) + \partial_\phi(\hat{q}^{12}(\phi)\partial_\rho u) + \partial_\phi(\hat{q}^{22}(\phi)\partial_\phi u) = 0,\\
&-\infty < \rho < 0, \quad \theta_i < \phi < \theta_{i+1},\\
&u(\rho, \theta_i^-) = u(\rho, \theta_i^+), \quad -\infty < \rho < 0,\\
&P_{i-1}\Big[-\bar{r}'\frac{\partial u}{\partial \rho} + \bar{r}\frac{\partial u}{\partial \phi}\Big]\Big|_{\phi=\theta_i^-} = P_i\Big[-\bar{r}'\frac{\partial u}{\partial \rho} + \bar{r}\frac{\partial u}{\partial \phi}\Big]\Big|_{\phi=\theta_i^+}, \quad -\infty < \rho < 0,\\
&u\Big|_{\rho=0} = f(\phi) = u_0(\phi), \quad u \text{ is bounded, as } \rho \to -\infty,
\end{aligned} \tag{2.12}$$

where $j = 1, \ldots, J$, $\theta_{J+1} = 2\pi$ and $P_0 = P_J$. We introduce $H^1(0, 2\pi) = \{v(\phi)|v(\phi), v'(\phi) \in L^2(0, 2\pi)\}$, $V = \{v(\phi) \in H^1(0, 2\pi)|v(0) = v(2\pi)\}$ and $U = \{u(\rho, \phi)|u, \partial_\rho u, \partial^2_{\rho\rho} u \in V \text{ for fixed } \rho < 0\}$. Then problem (2.12) is equivalent to the following variational–differential problem:

$$\begin{aligned}
&\text{Find } u(\rho, \phi) \in U \text{ such that}\\
&\frac{d^2}{d\rho^2}A_2(u, v) + \frac{d}{d\rho}A_1(u, v) + A_0(u, v) = 0, \quad \forall v \in V,\\
&u|_{\rho=0} = u_0(\phi), \quad u \text{ is bounded, as } \rho \to -\infty,
\end{aligned} \tag{2.13}$$

where

$$A_2(u, v) = \sum_{i=1}^{J-1}\int_{\theta_i}^{\theta_{i+1}} P_i\left(1 + \left(\frac{\bar{r}'(\phi)}{\bar{r}(\phi)}\right)^2\right)u(\rho, \phi)v(\phi)d\phi, \tag{2.14}$$

$$A_1(u, v) = \sum_{i=1}^{J-1}\int_{\theta_i}^{\theta_{i+1}} P_i\left[-\frac{\bar{r}'(\phi)}{\bar{r}(\phi)}v(\phi)\partial_\phi u(\rho, \phi) + \frac{\bar{r}'(\phi)}{\bar{r}(\phi)}u(\rho, \phi)v'(\phi)\right]d\phi, \tag{2.15}$$

$$A_0(u, v) = -\sum_{i=1}^{J-1}\int_{\theta_i}^{\theta_{i+1}} P_i\partial_\phi u(\rho, \phi)v'(\phi)d\phi. \tag{2.16}$$

Remark 2.1. The BVP (2.12) in the curvilinear coordinates is equivalent to the interface problem (2.1) since the transformation of coordinates from the polar coordinates to the curvilinear coordinates, $(r, \theta) = (e^\rho \bar{r}(\phi), \phi)$, is a smooth bijection. Thus problem (2.12) is well-posed with a unique solution which is 2π-periodic in ϕ.

Remark 2.2. We note that both $H^1(-\infty, 0)$ and V are Hilbert spaces and $\overline{H^1(-\infty, 0) \times V} \cong W$, where $W = \{w \in H^1[(-\infty, 0) \times (0, 2\pi)]|w(\rho, 0) = w(\rho, 2\pi)\}$. Thus the variational problem of (2.13) is equivalent to that of (2.12) due to Fubini's theorem, which means the variational–differential problem (2.13) is well-posed as well.

2.2. Anisotropic case

In this case, the coefficient P in the interface problem (2.1) changes from a scalar function to a second order symmetric tensor that is positive definite. We let

$$P = \begin{pmatrix} p_{11} & p_{12}\\ p_{21} & p_{22}\end{pmatrix},$$

where $p_{12} = p_{21}$ and P is a piecewise constant tensor function. We still assume that $P \equiv P_j$ in $\Omega_j, j = 1, \ldots, J$.

After similar computations to those in the isotropic case, we obtain

$$\nabla_1 \cdot (P\nabla_1 u) = (\partial_\rho + 2, \partial_\phi + 2\bar{r}'/\bar{r})E^{-T}PE^{-1}\nabla_2 u. \tag{2.17}$$

If we further let $(a^{ij}) = E^{-T}PE^{-1}$, we have

$$\nabla_1 \cdot (P\nabla_1 u) = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}a^{ij}\partial_j u), \tag{2.18}$$

where $\sqrt{g}$ is the same as in the isotropic case. With $\hat{a}^{ij} = \sqrt{g}a^{ij}$, we obtain

$$\sqrt{g}\nabla_1 \cdot (P\nabla_1 u) = \hat{a}^{11}(\phi)\partial_\rho^2 u + \partial_\rho(\hat{a}^{12}(\phi)\partial_\phi u) + \partial_\phi(\hat{a}^{12}(\phi)\partial_\rho u) + \partial_\phi(\hat{a}^{22}(\phi)\partial_\phi u), \tag{2.19}$$

where with $R(\phi) = \bar{r}'(\phi)/\bar{r}(\phi)$,

$$\hat{a}^{11} = p_{11}(R\sin(\phi)+\cos(\phi))^2 - 2p_{12}(R\cos(\phi)-\sin(\phi))(R\sin(\phi)+\cos(\phi)) + p_{22}(R\cos(\phi)-\sin(\phi))^2, \tag{2.20}$$

$$\hat{a}^{12} = \hat{a}^{21} = p_{12}(R\sin(2\phi)+\cos(2\phi)) - p_{11}\sin(\phi)(R\sin(\phi)+R\cos(\phi)) - p_{22}\cos\phi(R\cos(\phi)-\sin(\phi)), \tag{2.21}$$

$$\hat{a}^{22} = p_{11}\sin^2(\phi) - 2p_{12}\sin(\phi)\cos(\phi) + p_{22}\cos^2(\phi). \tag{2.22}$$

Then in the anisotropic case, the interface problem (2.1) can be formalized in the curvilinear coordinates as

$$\begin{cases} \hat{a}^{11}(\phi)\partial_\rho^2 u + \partial_\rho(\hat{a}^{12}(\phi)\partial_\phi u) + \partial_\phi(\hat{a}^{12}(\phi)\partial_\rho u) + \partial_\phi(\hat{a}^{22}(\phi)\partial_\phi u) = 0, \\ -\infty < \rho < 0, \quad \theta_i < \phi < \theta_{i+1}, \\ u(\rho, \theta_i^-) = u(\rho, \theta_i^+), \quad -\infty < \rho < 0, \\ \left[P_{i-1}E^{-1}\nabla_2 u \cdot \vec{n}\right]\Big|_{\phi=\theta_i^-} = \left[P_i E^{-1}\nabla_2 u \cdot \vec{n}\right]\Big|_{\phi=\theta_i^+}, \quad -\infty < \rho < 0, \\ u\Big|_{\rho=0} = f(\phi) = u_0(\phi), \quad u \text{ is bounded, as } \rho \to -\infty, \end{cases} \tag{2.23}$$

where $\vec{n} = (-\sin(\phi), \cos(\phi))$. Similarly, problem (2.23) can be further reduced to a variational–differential problem:

$$\begin{cases} \text{Find } u(\rho,\phi) \in U \text{ such that} \\ \dfrac{d^2}{d\rho^2}A_2(u,v) + \dfrac{d}{d\rho}A_1(u,v) + A_0(u,v) = 0, \quad \forall v \in V, \\ u|_{\rho=0} = u_0(\phi), \quad u \text{ is bounded, as } \rho \to -\infty, \end{cases} \tag{2.24}$$

where

$$A_2(u,v) = \int_0^{2\pi} \hat{a}^{11}(\phi)u(\rho,\phi)v(\phi)d\phi, \tag{2.25}$$

$$A_1(u,v) = \int_0^{2\pi} [\hat{a}^{12}(\phi)v(\phi)\partial_\phi u(\rho,\phi) - \hat{a}^{21}(\phi)u(\rho,\phi)v'(\phi)]d\phi, \tag{2.26}$$

$$A_0(u,v) = -\int_0^{2\pi} \hat{a}^{22}(\phi)\partial_\phi u(\rho,\phi)v'(\phi)d\phi. \tag{2.27}$$

It is easy to see that the above variational–differential problem would degenerate to that in the isotropic case if P is an isotropic tensor.

Lemma 2.1. *In both isotropic and anisotropic cases,*

1. $A_j(u,v)$ $(j = 0, 1, 2)$ *are three bounded bilinear forms on* $V \times V$*;*
2. $A_0(u,v)$ *and* $A_2(u,v)$ *are symmetric while* $A_1(u,v)$ *is antisymmetric;*
3. *there exists a constant* $\mu > 0$*, such that*

$$-A_0(v,v) \ge \mu|v|^2_{1,(0,2\pi)}, \quad A_2(v,v) \ge \mu\|v\|^2_{0,(0,2\pi)}, \quad \forall v \in V.$$

3. Numerical treatments to the variational–differential problem

We focus on the numerical treatments in the isotropic case. Suppose that $0 = \phi_1 < \phi_2 < \cdots < \phi_{M+1} = 2\pi$ is a partition of the interval $[0, 2\pi]$ such that each of $\{\theta_i \,|\, i = 1, 2, \ldots, J\}$ is a node of this partition, namely for θ_i, there is a $\phi_j = \theta_i$. Let $h = \max_{1\le j\le M}(\phi_{j+1} - \phi_j)$ and V_h be a finite dimensional subspace of V. Moreover, suppose that $U_h = \{u_h(\rho,\phi) | u_h, \partial_\rho u_h, \partial^2_{\rho\rho} u_h \in V_h \text{ for fixed } \rho < 0\}$. Then we have the semi-approximation of problem (2.13):

$$\begin{cases} \text{Find } u_h(\rho,\phi) \in U_h \text{ such that} \\ \dfrac{d^2}{d\rho^2}A_2(u_h,v_h) + \dfrac{d}{d\rho}A_1(u_h,v_h) + A_0(u_h,v_h) = 0, \quad \forall v_h \in V_h, \\ u_h|_{\rho=0} = u_{0,h}(\phi), \quad u_h \text{ is bounded, as } \rho \to -\infty, \end{cases} \tag{3.1}$$

where $u_{0,h}(\phi) \in V_h$ is the projection of $u_0(\phi)$ onto V_h. V_h can be any finite dimensional subspace of V. If the linear elements are used,

$$V_h = \left\{ v_h(\phi) \middle| \, v_h(\phi) \in V, \; v_h|_{[\phi_j,\phi_{j+1}]} \in P_1([\phi_j,\phi_{j+1}]), \; j = 1, 2, \ldots, M \right\}.$$

Assume that $\{N_j(\phi),\ j = 1, 2, \ldots, M\}$ is a basis of V_h and $N_j(\phi_i) = \delta_{ij}$, for $1 \le i, j \le M$. Let

$$N(\phi) = [N_1(\phi), N_2(\phi), \ldots, N_M(\phi)]^T.$$

For $u_h \in U_h$, let

$$U(\rho) \equiv [u_h(\rho, \phi_1), \ldots, u_h(\rho, \phi_M)]^T$$

and

$$U_0 \equiv [u_0(\phi_1), \ldots, u_0(\phi_M)]^T,$$

thus

$$u_h(\rho, \phi) = N(\phi)^T\, U(\rho),$$
$$u_{0,h}(\phi) = N(\phi)^T\, U_0.$$

Then the semi-approximation (3.1) is reduced to the following boundary value problem of a second-order ODE system:

$$\begin{aligned} &B_2U''(\rho) + B_1U'(\rho) + B_0U(\rho) = 0, \quad -\infty < \rho < 0,\\ &U(0) = U_0, \quad U(\rho) \text{ is bounded, as } \rho \to -\infty, \end{aligned} \tag{3.2}$$

where $B_j (j = 0, 1, 2)$ are three $M \times M$ constant coefficient matrices given by

$$B_2 = \sum_{i=1}^{J-1} \int_{\theta_i}^{\theta_{i+1}} P_i \left(1 + \left(\frac{\bar{r}'(\phi)}{\bar{r}(\phi)}\right)^2\right) N(\phi)N(\phi)^T d\phi, \tag{3.3}$$

$$B_1 = \sum_{i=1}^{J-1} \int_{\theta_i}^{\theta_{i+1}} -P_i \frac{\bar{r}'(\phi)}{\bar{r}(\phi)} [N(\phi)N'(\phi)^T - N'(\phi)N(\phi)^T] d\phi, \tag{3.4}$$

$$B_0 = -\sum_{i=1}^{J-1} \int_{\theta_i}^{\theta_{i+1}} P_i N'(\phi)N'(\phi)^T d\phi. \tag{3.5}$$

It is straightforward to compute B_j from the above expressions. From Lemma 2.1, we have

Lemma 3.1. *B_2 is a positive definite symmetric matrix, B_1 is an antisymmetric matrix and B_0 is a semi-negative definite symmetric matrix.*

We adopt a direct method for solving the boundary value problem of (3.2) on the interval $(-\infty, 0]$. Let

$$U(\rho) = e^{\rho\lambda}\xi, \tag{3.6}$$

where λ is a constant and $\xi \in \mathbb{C}^M$ is to be determined. Substituting (3.6) into the ODE system (3.2), we get the following quadratic eigenvalue problem:

$$[\lambda^2 B_2 + \lambda B_1 + B_0]\xi = 0. \tag{3.7}$$

If we let $\eta = \lambda\xi$, the quadratic eigenvalue problem (3.7) is further reduced to the following standard eigenvalue problem:

$$\begin{pmatrix} 0 & I_M \\ -B_0 & -B_1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} I_M & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{3.8}$$

where I_M denotes the $M \times M$ identity matrix.

From Lemma 3.1 and the results in [39], we know that the eigenvalues of (3.7) or (3.8) are purely imaginary or come in pairs $(\lambda, -\bar{\lambda})$. Thus we have the following lemma:

Lemma 3.2. *The eigenvalue problem (3.7) or (3.8) yields M eigenvalues with non-negative real parts while the other M eigenvalues have non-positive real parts.*

After solving the eigenvalue problem (3.8) numerically, we obtain the eigenvalues λ_j^h $(j = 1, 2, \ldots, M)$ with non-negative real parts corresponding to the eigenvectors:

$$\begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix} \qquad j = 1, 2, \ldots, M$$

where $\lambda_1^h = 0$, $\xi_1 = (1, \ldots, 1)^T \in \mathbb{R}^M$, $\eta_1 = (0, \ldots, 0)^T \in \mathbb{R}^M$. In particular, we assume that λ_j^h $(1 \le j \le \ell)$ are real eigenvalues, λ_j^h $(\ell + 1 \le j \le M)$ are complex eigenvalues with nonzero imaginary parts such that $\lambda_{2j}^h = \bar{\lambda}_{2j-1}^h$ $(\frac{\ell+2}{2} \le j \le \frac{M}{2})$ since B_j $(j = 0, 1, 2)$ are real matrices as well. (Without loss of generality, we assume that M is an even integer.) Hence we obtain

$$U(\rho) = \sum_{j=1}^{\ell} b_j e^{\rho\lambda_j^h} \xi_j + \sum_{j=\ell/2+1}^{M/2} \left[b_{2j-1}\mathbf{Re}(e^{\rho\lambda_{2j}^h}\xi_{2j}) + b_{2j}\mathbf{Im}(e^{\rho\lambda_{2j}^h}\xi_{2j}) \right], \tag{3.9}$$

satisfying the ODE system and boundary condition in (3.2). And $U(\rho)$ is bounded when $\rho \to -\infty$ for any $\{b_j\}_{j=1}^M$. By the boundary condition $U(0) = U_0$, we have

$$U_0 = \sum_{j=1}^{\ell} b_j \xi_j + \sum_{j=\ell/2+1}^{M/2} \left[b_{2j-1} \mathbf{Re}(\xi_{2j}) + b_{2j} \mathbf{Im}(\xi_{2j}) \right], \tag{3.10}$$

where $\mathbf{Re}(\xi_{2j})$ and $\mathbf{Im}(\xi_{2j})$ denote the real part and imaginary part of ξ_{2j} respectively. We introduce the following matrices:

$$D(\rho) = \left[e^{\rho\lambda_1^h}\xi_1, \ldots, e^{\rho\lambda_\ell^h}\xi_\ell, \mathbf{Re}(e^{\rho\lambda_{\ell+2}^h}\xi_{\ell+2}), \mathbf{Im}(e^{\rho\lambda_{\ell+2}^h}\xi_{\ell+2}), \ldots, \mathbf{Re}(e^{\rho\lambda_M^h}\xi_M), \mathbf{Im}(e^{\rho\lambda_M^h}\xi_M) \right],$$

$$D_0 \equiv D(0) = [\xi_1, \ldots, \xi_\ell, \mathbf{Re}(\xi_{\ell+2}), \mathbf{Im}(\xi_{\ell+2}), \ldots, \mathbf{Re}(\xi_M), \ \mathbf{Im}(\xi_M)],$$

$$B = [b_1, b_2 \ldots, b_M].$$

Then (3.10) can be expressed in the matrix form as $U_0 = D_0 B$. Thus we have

$$B = D_0^{-1} U_0. \tag{3.11}$$

Substituting (3.11) into (3.9), we obtain the solution of the second-order ODE system (3.2),

$$U(\rho) = D(\rho) D_0^{-1} U_0. \tag{3.12}$$

We eventually get the semi-discrete approximate solution of problem (2.1):

$$u_h(\rho, \phi) = N(\phi)^T D(\rho) D_0^{-1} U_0. \tag{3.13}$$

Remark 3.1. To introduce the basic idea and implementation of our generalized direct method of lines, we state how to solve an interface problem using our method in Sections 2 and 3. However, our method is not limited to solving interface problems and can be applied to other elliptic problems as well. For example, it can be applied to the boundary value problem in corner domains [35], where some part of domain is removed. The corresponding numerical example is Example 4.2. What is more, it can also be applied to the boundary value problem outside a bounded domain [37,38]. The corresponding numerical example is Example 4.3. To apply our method to these problems, the idea and implementation are similar. We just need some slight modifications. And the numerical examples will show that our method is effective for these problems.

Remark 3.2. In the above process, we choose the eigenvalues with non-negative real parts when constructing $U(\rho)$ due to the boundary condition that $U(\rho)$ is bounded as $\rho \to -\infty$. However, when dealing with the exterior problem [37,38], the corresponding boundary condition would be $U(\rho)$ is bounded as $\rho \to +\infty$ and hence we should choose the M eigenvalues with non-positive real parts. And Lemma 3.2 ensures the feasibility of the implementation.

Remark 3.3. It is possible to generalize our method to study the elliptic problem with a variable coefficient $P = P(x)$ in some cases. If P is only dependent on the angular variable ϕ, i.e. $P = P(\phi)$, the generalization is straightforward. We can obtain the variational–differential problem in the same form as Eq. (2.13) and the second-order ODE system in the same form as Eq. (3.2) with a slight modification in the expressions of A_2, A_1, A_0, B_2, B_1 and B_0. Take B_0 for example. We have

$$B_0 = -\int_0^{2\pi} P(\phi) N'(\phi) N'(\phi)^T d\phi$$

instead of

$$B_0 = -\sum_{i=1}^{J-1} \int_{\theta_i}^{\theta_{i+1}} P_i N'(\phi) N'(\phi)^T d\phi.$$

The modifications for other coefficients are similar. The subsequent implementation of the method remains the same. Such a generalization is also valid for the anisotropic case.

But if P has a non-trivial dependence on ρ, i.e. $P = P(\rho, \phi)$ and $\partial_\rho P(\rho, \phi) \not\equiv 0$, the generalization is not that easy. This remains a research topic for us in the future.

We now turn to the error estimate of our numerical treatment. We still focus on the interface problem (2.1) in the isotropic case. We want to estimate the error $\|u - u_h\|_{1,\Omega}$, where u is the exact solution of problem (2.1) and u_h is the numerical solution of the discrete variational–differential problem (3.1). Note that $\bar{r}(\phi)$ is smooth, 2π-periodic and that $\bar{r}(\phi) \geq r_0 > 0$, $\forall \phi \in [0, 2\pi]$, which indicates that $\hat{q}^{ij}$ ($\hat{a}^{ij}$ respectively in the anisotropic case) are bounded for $i, j = 1, 2$, we have the following theorem:

Theorem 3.3. *If the linear elements are used, i.e.,*

$$V_h = \left\{ v_h(\phi) \middle| \ v_h(\phi) \in V, \ v_h|_{[\phi_j, \phi_{j+1}]} \in P_1([\phi_j, \phi_{j+1}]), \ j = 1, 2, \ldots, M \right\},$$

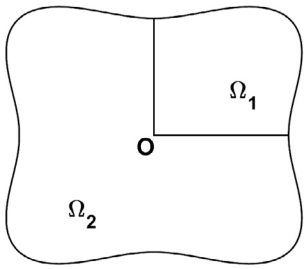

Fig. 2. Interface problem.

then the following estimate holds

$$\|u-u_h\|^2_{1,\Omega} \le Ch^2 \sum_{i=1}^{J} \left\{ \iint_{\Omega_i} \left[|\nabla u|^2 + \sum_{|\alpha|=2} |D^\alpha u|^2 \right] dxdy + \|f\|^2_{1,\Gamma_i} \right\}, \tag{3.14}$$

where C is a positive constant independent of h and $\Gamma_i = \partial\Omega_i \bigcap \Gamma$.

The idea of proofs of the above theorem is similar to that in [35]. So we omit the proofs here. For detailed proofs, please refer to Section 4 in [35].

Remark 3.4. In general, the quadratic elements or other higher-order elements can be used to construct the subspace V_h. And similarly we would get a semi-discrete approximate solution of problem (2.1), which would possess higher accuracy. Our numerical examples will show this fact.

4. Numerical examples

We give four numerical examples in this section.

Example 4.1. Interface problem.

Let $\bar{r}(\theta) = \frac{1}{\sqrt{\cos^4(\theta)+\sin^4(\theta)}}$ and $\Omega = \{(r,\theta)|0 \le r < \bar{r}(\theta), 0 \le \theta \le 2\pi\}$, $\Omega_1 = \{(r,\theta)|0 < r < \bar{r}(\theta), 0 < \theta < \frac{\pi}{2}\}$, $\Omega_2 = \{(r,\theta)|0 < r < \bar{r}(\theta), \frac{\pi}{2} < \theta < 2\pi\}$ (see Fig. 2), and

$$P = \begin{cases} 10, & \text{on } \Omega_1, \\ 1, & \text{on } \Omega_2, \end{cases}$$
$$f = r^{\lambda_2} h_2(\theta) + r^{\lambda_3} h_3(\theta),$$

where

$$\lambda_2 = 1-a, \qquad \lambda_3 = 1+a, \qquad a = 2\eta/\pi,\ \eta = \arcsin 0.9/2.2,$$
$$h_2(\theta) = \begin{cases} \cos(1-a)\theta + C_3 \sin(1-a)\theta, & 0 \le \theta \le \pi/2, \\ C_1\cos(1-a)\theta + C_2C_3\sin(1-a)\theta, & \pi/2 \le \theta \le 2\pi, \end{cases}$$
$$h_3(\theta) = \begin{cases} \cos(1+a)\theta - C_3 \sin(1+a)\theta, & 0 \le \theta \le \pi/2, \\ C_1\cos(1+a)\theta - C_2C_3\sin(1+a)\theta, & \pi/2 \le \theta \le 2\pi, \end{cases}$$

with

$$C_1 = (1 + 2\sin\eta - 4\sin^2\eta)/(1-2\sin\eta),$$
$$C_2 = (1 - 2\sin\eta - 4\sin^2\eta)/(1-2\sin\eta),$$
$$C_3 = (1-\sin\eta)/\cos\eta.$$

Consider the following interface problem:

$$\begin{aligned} &-\nabla\cdot(P\nabla u) = 0, \quad \text{in } \Omega, \\ &u|_\Gamma = f|_\Gamma, \\ &u(r,\theta_j^-) = u(r,\theta_j^+), \quad 0 \le r \le \bar{r}(\theta_j), \\ &P(\theta_j^-)u_\theta(r,\theta_j^-) = P(\theta_j^+)u_\theta(r,\theta_j^+), \quad 0 < r < \bar{r}(\theta_j), \end{aligned} \tag{4.1}$$

where $j = 1, 2$ and $\theta_1 = 0, \theta_2 = \pi/2$.

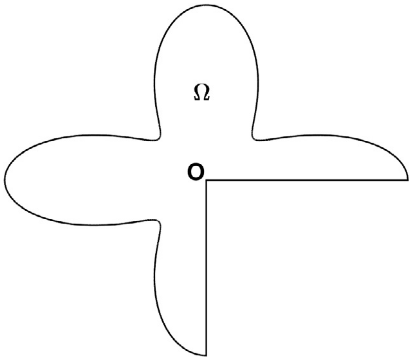

Fig. 3. Problem with corner point.

Table 1
Results of Example 4.1. Linear elements.

M	$\lvert\lambda_2^h-\lambda_2\rvert$	$\lvert\lambda_3^h-\lambda_3\rvert$	$\lVert u-u_h\rVert_{0,\Omega}/\lVert u\rVert_{0,\Omega}$	$\lVert u-u_h\rVert_{1,\Omega}/\lVert u\rVert_{1,\Omega}$
16	1.54210e−02	2.74818e−02	3.66939e−02	1.29410e−01
32	4.97451e−03	9.22507e−03	1.13096e−02	7.52660e−02
64	1.28886e−03	2.42114e−03	2.90167e−03	3.64710e−02
128	3.25086e−04	6.12673e−04	7.29890e−04	1.80242e−02
256	8.14527e−05	1.53634e−04	1.82753e−04	8.98434e−03

Table 2
Results of Example 4.1. Quadratic elements.

M	$\lvert\lambda_2^h-\lambda_2\rvert$	$\lvert\lambda_3^h-\lambda_3\rvert$	$\lVert u-u_h\rVert_{0,\Omega}/\lVert u\rVert_{0,\Omega}$	$\lVert u-u_h\rVert_{1,\Omega}/\lVert u\rVert_{1,\Omega}$
16	1.80627e−03	4.22185e−03	6.69986e−03	4.46586e−02
32	9.08411e−05	2.36297e−04	7.70916e−04	9.95890e−03
64	5.77827e−06	1.51067e−05	9.84948e−05	2.44789e−03
128	3.63606e−07	9.50718e−07	1.24351e−05	6.11639e−04
256	2.27638e−08	5.95221e−08	1.55829e−06	1.52888e−04

Table 3
Results of Example 4.2. Linear elements.

M	$\lvert\lambda_2^h-\lambda_2\rvert$	$\lvert\lambda_3^h-\lambda_3\rvert$	$\lVert u-u_h\rVert_{0,\Omega}/\lVert u\rVert_{0,\Omega}$	$\lVert u-u_h\rVert_{1,\Omega}/\lVert u\rVert_{1,\Omega}$
16	9.88089e−02	2.17913e−01	5.86646e−02	2.23902e−01
32	2.85187e−02	6.47301e−02	1.56697e−02	9.96356e−02
64	7.54962e−03	1.68563e−02	3.98375e−03	4.82895e−02
128	1.91545e−03	4.25628e−03	1.00027e−03	2.39637e−02
256	4.80643e−04	1.06670e−03	2.53549e−04	1.19597e−02

Then $u(r,\theta)=r^{\lambda_2}h_2(\theta)+r^{\lambda_3}h_3(\theta)$ is the unique solution of (4.1). (See Tables 1 and 2.)

Example 4.2. Corner singularities.
Let $\bar{r}(\theta)=2+\cos(4\theta)$ and

$$\Omega=\{(r,\theta)|0<r<\bar{r}(\theta),0<\theta<3\pi/2\},$$

$$\Gamma_N=\{(r,\theta)|0\le r\le\bar{r}(\theta),\theta=0\text{ or }\theta=3\pi/2\},$$

$$\Gamma_D=\partial\Omega\setminus\Gamma_N.$$

(See Fig. 3). Consider the following boundary value problem:

$$\begin{cases}-\Delta u=0 \quad \text{in } \Omega,\\ u|_{\Gamma_D}=f|_{\Gamma_D},\\ \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma_N}=0,\end{cases} \tag{4.2}$$

where $f(r,\theta)=r^{2/3}\cos(\frac{2}{3}\theta)+r^{4/3}\cos(\frac{4}{3}\theta)$. Then $u(r,\theta)=f(r,\theta)$ is the unique solution of (4.2). (See Tables 3 and 4.)

Table 4
Results of Example 4.2. Quadratic elements.

M	$\|\lambda_2^h - \lambda_2\|$	$\|\lambda_3^h - \lambda_3\|$	$\|u-u_h\|_{0,\Omega}/\|u\|_{0,\Omega}$	$\|u-u_h\|_{1,\Omega}/\|u\|_{1,\Omega}$
16	2.57504e−03	7.21007e−03	4.81530e−03	9.46886e−02
32	3.08120e−04	4.21603e−04	6.32814e−04	2.43882e−02
64	1.96655e−05	2.55890e−05	8.16560e−05	6.15971e−03
128	1.23471e−06	1.58481e−06	1.03391e−05	1.54639e−03
256	7.72584e−08	9.88313e−08	1.29943e−06	3.87349e−04

Table 5
Results of Example 4.3. Linear elements.

M	$\|u-u_h\|_{0,\Omega}/\|u\|_{0,\Omega}$	$\|u-u_h\|_{1,\Omega}/\|u\|_{1,\Omega}$
16	1.81332e−02	6.51841e−02
32	4.63327e−03	3.08164e−02
64	1.16450e−03	1.50819e−02
128	2.91515e−04	7.49850e−03
256	7.29033e−05	3.74401e−03

Table 6
Results of Example 4.3. Quadratic elements.

M	$\|u-u_h\|_{0,\Omega}/\|u\|_{0,\Omega}$	$\|u-u_h\|_{1,\Omega}/\|u\|_{1,\Omega}$
16	1.32291e−03	9.30685e−03
32	1.69012e−04	2.26967e−03
64	2.13271e−05	5.64701e−04
128	2.67247e−06	1.41004e−04
256	3.34266e−07	3.52402e−05

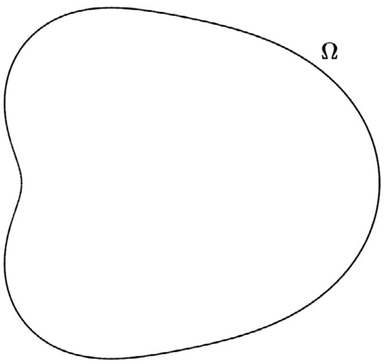

Fig. 4. Exterior problem.

Example 4.3. Exterior problem.

Let $\bar{r}(\theta) = 2 + \cos^3(\theta)$ and $\Omega = \{(r,\theta)|r > \bar{r}(\theta), 0 \le \theta \le 2\pi\}$ (see Fig. 4). Consider the following exterior problem:

$$\begin{aligned} &-\Delta u = 0 \quad \text{in } \Omega, \\ &u|_\Gamma = f|_\Gamma, \\ &u \text{ is bounded as } r \to +\infty, \end{aligned} \tag{4.3}$$

where $f(r,\theta) = \frac{1}{2}\frac{r^2 + r\sin(\theta) + 1/4}{r^2 - r\sin(\theta) + 1/4}$. It is straightforward to check that $u(r,\theta) = f(r,\theta)$ is the unique solution of (4.3). (See Tables 5 and 6.)

Example 4.4. Anisotropic case.

Let $\bar{r}(\theta) = \sqrt{\frac{3}{2} + \sin(5\theta)}$ and $\Omega = \{(r,\theta)|0 \le r < \bar{r}(\theta), 0 \le \theta \le 2\pi\}$ (see Fig. 5),

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

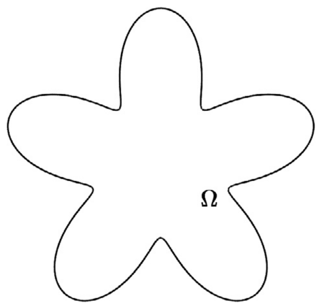

Fig. 5. Anisotropic problem.

Table 7
Results of Example 4.4. Linear elements.

M	$\|u-u_h\|_{0,\Omega}/\|u\|_{0,\Omega}$	$\|u-u_h\|_{1,\Omega}/\|u\|_{1,\Omega}$
128	3.81037e−03	4.13534e−02
256	9.94178e−04	2.05363e−02
512	2.57196e−04	1.02471e−02
1024	6.53754e−05	5.12011e−03

Table 8
Results of Example 4.4. Quadratic elements.

M	$\|u-u_h\|_{0,\Omega}/\|u\|_{0,\Omega}$	$\|u-u_h\|_{1,\Omega}/\|u\|_{1,\Omega}$
128	6.25065e−05	1.71809e−03
256	7.80882e−06	4.04984e−04
512	9.76104e−07	1.01210e−04
1024	1.27413e−07	2.53076e−05

Consider the following problem:

$$\begin{aligned} &-\nabla\cdot(P\nabla u)=0 \quad \text{in } \Omega,\\ &u|_\Gamma = f|_\Gamma, \end{aligned} \tag{4.4}$$

where $f(r,\theta)=r^2(\cos(2\theta)-\sin(\theta)\cos(\theta))$. Then $u(r,\theta)=f(r,\theta)$ is the unique solution of (4.4) (see Tables 7 and 8).

All the results show that our method has the first-order convergence rate for the linear elements and second-order for the quadratic elements w.r.t. the error in H^1-norm and the convergence rate is one order higher when considering the error in L^2-norm and the error of eigenvalues. These show that our method is feasible for a variety of elliptic problems in star-shaped domains.

5. Conclusion

In this paper, we generalize the direct method of lines for elliptic problems in star-shaped domains. We assume that the boundary of the domain is a closed curve that can be parameterized as a (piecewise) smooth function of the angular variable, which enables us to introduce an appropriate transformation of coordinates. Then the elliptic problem is reduced to a variational–differential problem, the well-posedness of which has been discussed. This variational–differential problem is only discretized w.r.t. the angular variable so that we obtain a semi-approximation. A direct method is introduced to solve it. We show that some good properties of the semi-approximation ensure that we can handle both interior and exterior problems and our method has an optimal error estimate. Finally, numerical results show that our method is feasible and effective for a wide range of elliptic problems.

References

[1] I. Babuska, Solution of problems with interfaces and singularities, Tech. Note. BN-789, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, 1974.
[2] I. Babuska, Singularities problem in the finite element method, Tech. Note. BN-835, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, 1976.
[3] R.B. Kellogg, Singularities in interface problems, in: Numerical Solution of Partial Differential Equations, Vol. 2, Academic Press, New York, 1971.
[4] R.B. Kellogg, On the Poisson equation with intersecting interfaces, Appl. Anal. 4 (1974) 101–129.
[5] T. Apel, S. Nicaise, The finite element method with anisotropic mesh grading for elliptic problems in domains with corners and edges, Math. Methods Appl. Sci. 21 (1998) 519–549.

[6] P. Grisvard, Elliptic Problems in Nonsmooth Domains, SIAM, 2011.
[7] K. Khan, C.S. Upadhyay, Exponentially accurate nonconforming least-squares spectral element method for elliptic problems on unbounded domain, Comput. Methods Appl. Mech. Engrg. 305 (2016) 607–633.
[8] S.V. Tsynkov, Numerical solution of problems on unbounded domains. A review, Appl. Numer. Math. 27 (1998) 465–532.
[9] I. Babuska, M.B. Rosenzweig, A finite element scheme for domains with corners, Numer. Math. 20 (1972) 1–21.
[10] I. Babuska, R.B. Kellogg, J. Pitkaranta, Direct and inverse error estimates for finite elements with mesh refinements, Numer. Math. 33 (1979) 447–471.
[11] H. Han, The numerical solutions of interface problems by infinite element method, Numer. Math. 39 (1982) 39–50.
[12] R.W. Thatcher, The use of infinite grid refinements at singularities in the solution of Laplace's equation, Numer. Math. 25 (1976) 163–178.
[13] I. Babuska, H.S. Oh, The p-version of the finite element method for domains with corners and for infinite domains, Numer. Methods Partial Differential Equations 6 (1990) 371–392.
[14] H. Han, The plane elastic problem with a boundary crack, Acta Sci. Natur. Univ. Pekinensis 2 (1983) 11–25.
[15] H.S. Oh, I. Babuska, The p-version of the finite element method for the elliptic boundary value problems with interfaces, Comput. Methods Appl. Mech. Engrg. 97 (1992) 211–231.
[16] Z. Cai, S. Kim, A finite element method using singular functions for the Poisson equation: Corner singularities, SIAM J. Numer. Anal. 39 (2001) 286–299.
[17] H.S. Oh, J.W. Jeong, Reproducing polynomial (singularity) particle methods and adaptive meshless methods for two-dimensional elliptic boundary value problems, Comput. Methods Appl. Mech. Engrg. 198 (2009) 933–946.
[18] C.C. Chu, I. Graham, T.Y. Hou, A new multiscale finite element method for high-contrast elliptic interface problems, Math. Comp. 79 (2010) 1915–1955.
[19] Z. Cai, X. Ye, S. Zhang, Discontinuous Galerkin finite element methods for interface problems: a priori and a posteriori error estimations, SIAM J. Numer. Anal. 49 (2011) 1761–1787.
[20] H. Egger, U. Rüde, B. Wohlmuth, Energy-corrected finite element methods for corner singularities, SIAM J. Numer. Anal. 52 (2014) 171–193.
[21] J. Helsing, R. Ojala, Corner singularities for elliptic problems: Integral equations, graded meshes, quadrature, and compressed inverse preconditioning, J. Comput. Phys. 227 (2008) 8820–8840.
[22] R.J. Leveque, Z. Li, The immersed interface method for elliptic equations with discontinuous coefficients and singular sources, SIAM J. Numer. Anal. 31 (1994) 1019–1044.
[23] Z. Li, K. Ito, The Immersed Interface Method: Numerical Solutions of PDEs Involving Interfaces and Irregular Domains, SIAM, 2006.
[24] Q. Zhang, K. Ito, Z. Li, Z. Zhang, Immersed finite elements for optimal control problems of elliptic PDEs with interfaces, J. Comput. Phys. 298 (2015) 305–319.
[25] L. Zhu, Z. Li, Z. Zhang, An immersed finite volume element method for 2D PDEs with discontinuous coefficients and non-homogeneous jump conditions, Comput. Math. Appl. 70 (2015) 89–103.
[26] B. Engquist, A. Majda, Absorbing boundary conditions for numerical simulation of waves, Math. Comp. 31 (1977) 629–651.
[27] R. Huan, L.L. Thompson, Accurate radiation boundary conditions for the time-dependent wave equation on unbounded domains, Internat. J. Numer. Methods Engrg. 47 (2000) 1569–1603.
[28] H. Han, X. Wu, Approximation of infinite boundary condition and its application to finite element methods, J. Comput. Math. 3 (1985) 179–192.
[29] H. Han, X. Wu, The approximation of the exact boundary conditions at an artificial boundary for linear elastic equations and its applications, Math. Comp. 59 (1992) 21–37.
[30] H. Han, Z. Huang, D. Yin, Exact artificial boundary conditions for quasilinear elliptic equations in unbounded domains, Commun. Math. Sci. 6 (2008) 71–82.
[31] T. Hagstrom, H.B. Keller, Exact boundary conditions at an artificial boundary for partial differential equations in cylinders, SIAM J. Math. Anal. 17 (1986) 322–341.
[32] T.M. Hagstrom, H.B. Keller, Asymptotic boundary conditions and numerical methods for nonlinear elliptic problems on unbounded domains, Math. Comp. 48 (1987) 449–470.
[33] W.E. Schiesser, The Numerical Method of Lines: Integration of Partial Differential Equations, Elsevier, 2012.
[34] L.S. Xanthis, C. Schwab, The method of arbitrary lines, C.R. Acad. Sci. Ser. I Math. 312 (1991) 181–187.
[35] H. Han, Z. Huang, The direct method of lines for the numerical solutions of interface problem, Comput. Methods Appl. Mech. Engrg. 171 (1999) 61–75.
[36] H. Han, Z. Huang, The discrete method of separation of variables for composite material problems, Int. J. Fract. 112 (2001) 379–402.
[37] H. Han, W. Bao, The discrete artificial boundary condition on a polygonal artificial boundary for the exterior problem of Poisson equation by using the direct method of lines, Comput. Methods Appl. Mech. Engrg. 179 (1999) 345–360.
[38] Z. Huang, High accuracy numerical method of thin-film problems in micromagnetics, J. Comput. Math. 21 (2003) 33–40.
[39] F. Tisseur, K. Meerbergen, The quadratic eigenvalue problem, SIAM Rev. 43 (2001) 235–286.