Rm $\displaystyle\int_C \vec{F}\cdot\vec{T}\,ds = \int_C M\,dx + N\,dy$

C: from (a,b) to (c,d) $\begin{cases} y = f(x) \\ x = g(y) \end{cases}$

$$\overset{?}{=} \int_a^c M\,dx + \int_b^d N\,dy$$

$$M = x+y,\quad N = x-y$$

Ans $\displaystyle = \int_a^c \underset{\substack{\| \\ M(x,f(x))}}{(x+f(x))}\,dx + \int_c^d \underset{\substack{\| \\ N(g(y),y)}}{(g(y)-y)}\,dy$

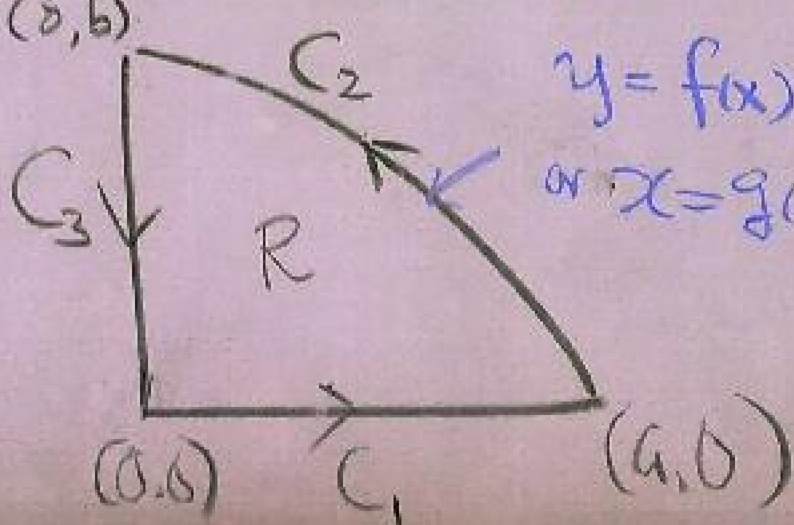
(take $t=x$) (take $t=y$)

Green's Thm : $\vec{F} = (M, N)$

[Assumptions]

$$\Rightarrow \begin{cases} \oint_C \vec{F} \cdot \vec{n} \, ds = \iint_R M_x + N_y \, dA \\ \oint_C \vec{F} \cdot \vec{T} \, ds = \iint_R N_x - M_y \, dA \end{cases}$$

Pf It suffices to check



Check tangential form

$$\iint_R N_x \, dA = \int_{y=0}^{b}\int_{x=0}^{g(y)} N_x \, dx \, dy$$

$$= \int_{y=0}^{b} \big(N(g(y),y) - N(0,y) \big) dy$$

$$= \int_0^b N(g(y),y)\, dy - \int_0^b N(0,y)\, dy$$

$$= \int_0^b (0,N)\cdot \overbrace{\left(\frac{dx}{dy}, \frac{dy}{dy}\right)}^{= \frac{d\vec{r}}{dt} dt,\ t=y} dy \Bigg|_{x=g(y)} = \int_{C_2} (0,N)\cdot \vec{T}\, ds$$

$$- \left(\int_0^b (0, N(0,y))\cdot(0,1)\, dy \right) = -\left(- \int_{C_3} (0,N)\cdot \vec{T}\, ds \right)$$

$$= - \int_{-C_3} (0,N)\cdot \vec{T}\, ds$$

Also $\int_{C_1} (0,N)\cdot \vec{T}\, ds = \int_{x=0}^{a} (0,N)\cdot(1,0)\, dx = 0$

$$\therefore \iint_R N_x \, dA = \int_{C_1+C_2+C_3} (0, N) \cdot \vec{T} \, ds$$

Similarly $$\iint_R -M_y \, dA = \int_{C_1+C_2+C_3} (M, 0) \cdot \vec{T} \, ds$$

$$\Rightarrow \iint_R (N_x - M_y) \, dA = \int_{C_1+C_2+C_3} (M, N) \cdot \vec{T} \, ds$$

Example: $\oint_C \vec{F} \cdot \vec{T}\, ds$

$\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) = (M, N)$

$C: \ x^2+y^2 = a^2$

Component test: $M_y = N_x$

$\text{Ans} = 2\pi$

But Green's Thm:

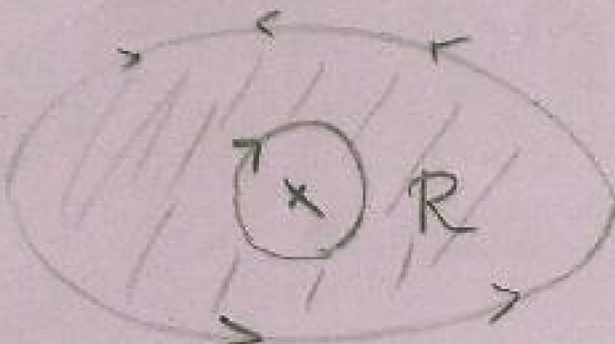
$\Rightarrow \text{Ans} = \iint_{R=\{x^2+y^2<a^2\}} N_x - M_y \, dA = 0$

What went wrong?

<u>Ans</u>: M, N and first derivatives are <u>not</u> cont. in R $\not\Rightarrow$ Green's Thm

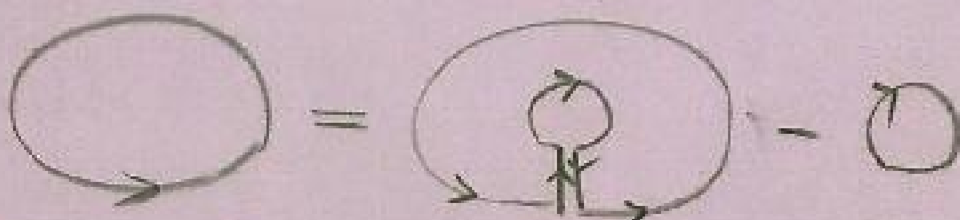
Example Same $\vec{F}$, $\oint \vec{F}\cdot\vec{T}\,ds = ?$

$C: \frac{x^2}{4}+y^2=1$



Let $R = \left\{ x^2+y^2>a^2,\ \frac{x^2}{4}+y^2<1 \right\}$ $(a<1)$

Ans



In other words

Boundary of R =

$$\iint_R N_x - M_y\, dA = \oint_{\frac{x^2}{4}+y^2=1} \vec{F}\cdot\vec{T}\,ds - \oint_{x^2+y^2=a^2} \vec{F}\cdot\vec{T}\,ds$$

$$\therefore\ 0 = \oint_{\frac{x^2}{4}+y^2=1} \vec{F}\cdot\vec{T}\,ds - 2\pi$$

Ans $= 2\pi$

<u>Rm</u> For $\vec{F} = \frac{(-y,x)}{x^2+y^2}$

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \begin{cases} 2\pi, & (0,0)\in R \\ 0, & (0,0)\notin R \end{cases}$$

R = inside of C

[<u>Rm</u> $\frac{(-y,x)}{x^2+y^2} = \nabla \tan^{-1}\left(\frac{y}{x}\right)$

But $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is NOT everywhere defined on $x^2+y^2\neq 0$]