Let $\vec{F} = (M, N, P)$

M, N, P and their first derivative are cont. in D.

$$\left\{ \begin{array}{l} \displaystyle\int_A^B \vec{F}\cdot\vec{T}\,ds \text{ is path indep.} \\ \Leftrightarrow \displaystyle\oint_C \vec{F}\cdot\vec{T}\,ds = 0 \\ \quad \text{for all closed curve } C \text{ in } D \\ \Leftrightarrow \vec{F} = \nabla f \end{array} \right.$$

$$\begin{array}{l} \Longrightarrow \\ \not\Longleftarrow (*) \end{array} \left\{ \begin{array}{l} M_y = N_x \\ N_z = P_y \\ P_x = M_z \end{array} \right.$$

$\overset{(*)}{\Longleftarrow}$ holds if

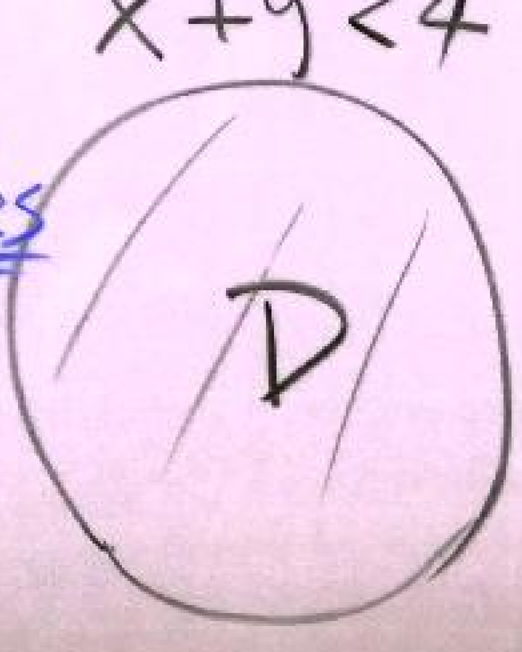
D is simply connected

Def D is simply connected if every closed loop in D can be contracted to a point without leaving D.

Examples Simply connected?

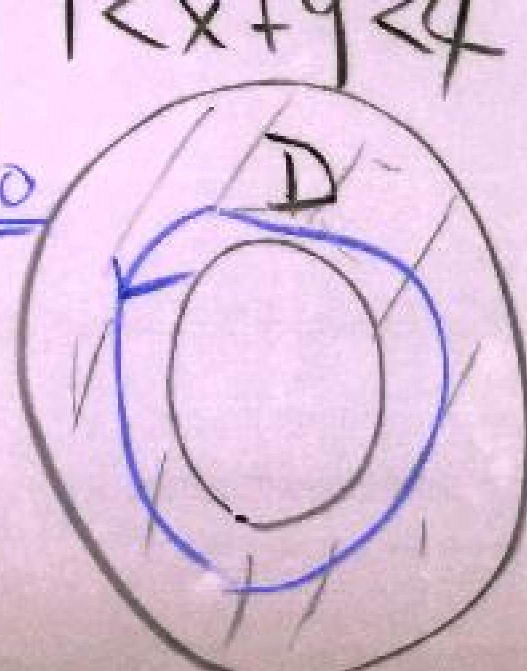
$x^2+y^2<4$

Yes



$1<x^2+y^2<4$

No

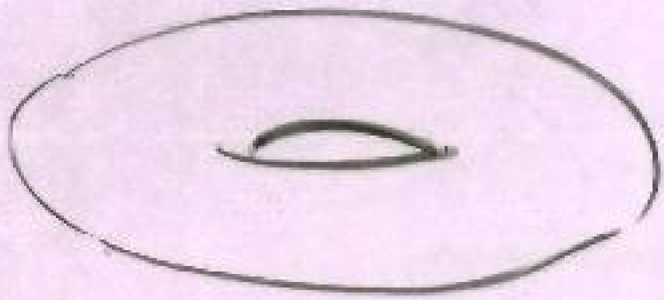


$x^2+y^2+z^2<4 \rightarrow$ Yes

$\begin{cases} 1<x^2+y^2+z^2<4 \rightarrow \text{yes} \\ \mathbb{R}^3 \setminus (0,0,0) \rightarrow \text{yes} \end{cases}$

$1<x^2+y^2<4$ in $\mathbb{R}^3 \rightarrow$ No

$\mathbb{R}^3 \setminus \{z \text{ axis}\} \rightarrow$ No

 $\rightarrow$ No

$(\sqrt{x^2+y^2}-2)^2+z^2<1$

If D is not simply connected

$\overset{(*)}{\Longleftarrow}$ may or may not

be true, depending on $\vec{F}$

Example $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$

Component test:

$$M_y = \frac{-1}{x^2+y^2} + \frac{y \cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$N_x = \frac{1}{x^2+y^2} + \frac{x(-2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\underline{M_y = N_x}$

Note: $\vec{F}$ is defined on its natural domain $\{x^2+y^2 \neq 0\}$

not simply connected

C: $\vec{r}(t) = (r\cos t, r\sin t, 0)$

$r > 0$, fixed $0 \leq t \leq 2\pi$

$$\oint_C \vec{F} \cdot \vec{T} \, ds = ?$$

$$\vec{T} = \frac{(-r\sin t, r\cos t, 0)}{r} = (-\sin t, \cos t, 0)$$

$$ds = \left|\frac{d\vec{r}}{dt}\right| dt = r\,dt$$

$$Ans = \int_0^{2\pi} \left(\frac{-r\sin t}{r^2}, \frac{r\cos t}{r^2}, 0\right)(-\sin t, \cos t, 0)\, r\,dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi \neq 0$$

<u>Example</u> $\vec{F} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0\right)$

Natural domain $= \{x^2+y^2 \neq 0\}$

not simply connected

But $\vec{F} = \nabla f = \nabla\left(\frac{\sqrt{x^2+y^2}}{2}\right)$

Green's Theorem

$\vec{F} = (M, N)$

M, N and first derivative are all continuous.

C: Simple closed curve.

R = interior of C

Then

(i) $\oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C M\,dy - N\,dx = \iint_R (M_x + N_y)\, dA$

(normal form)

(ii) $\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$

(tangential form)

Rm $(M, N) \longleftrightarrow (N, -M)$

(normal form) $\longleftrightarrow$ (tangential form)

i.e. normal form $\Longleftrightarrow$ tangential form

Rm These are special cases of

$$\iiint_D \underset{(\mathrm{div}\,\vec{F})}{\nabla \cdot \vec{F}}\, dV = \iint_{\partial D} \vec{F} \cdot \vec{n}\, d\sigma$$

$$\iint_S \underset{(\mathrm{curl}\,\vec{F})}{(\nabla \times \vec{F})} \cdot \vec{n}\, d\sigma = \int_{\partial S} \vec{F} \cdot \vec{T}\, ds$$

$d\sigma$: surface integral

∂D: boundary of D

∂S: boundary of S

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} \quad \left(\simeq (\partial_x, \partial_y, \partial_z)\cdot(F_1, F_2, F_3)\right)$$

$$= \partial_x F_1 + \partial_y F_2 + \partial_z F_3$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} \quad \left(\simeq (\partial_x, \partial_y, \partial_z)\times(F_1, F_2, F_3)\right)$$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

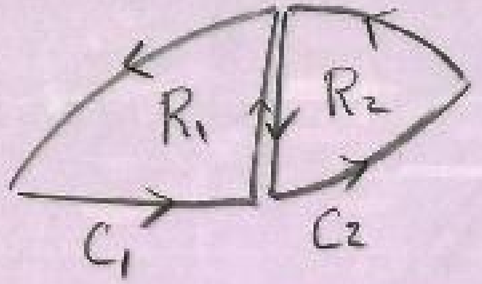
In 2D case

$$\underline{\operatorname{div}.}\ \vec{F}(x,y,0) = \partial_x F_1 + \partial_y F_2 = M_x + N_y$$

$$(\operatorname{curl}\vec{F})\cdot\vec{k} = \text{3rd component of } \nabla\times\vec{F}$$

$$= \partial_x F_2 - \partial_y F_1 = N_x - M_y$$

Pf. (tangential form)

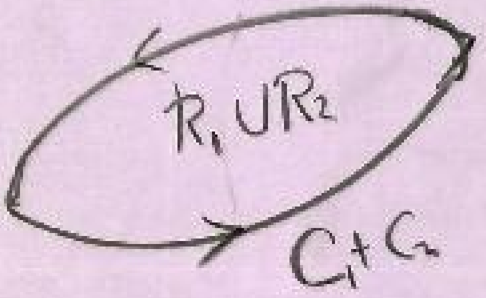


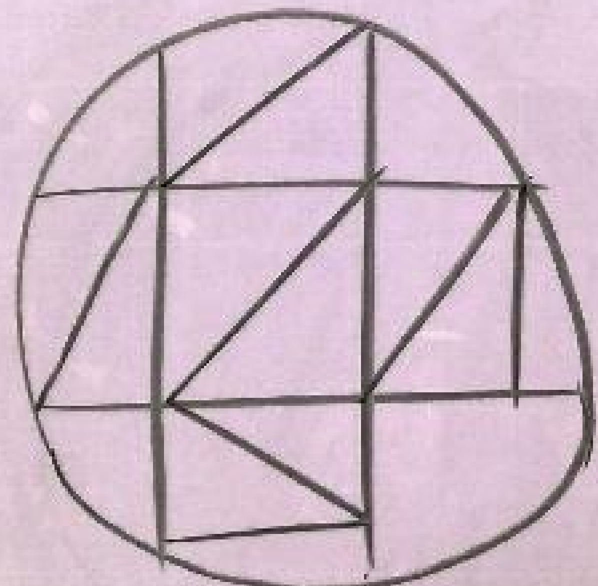
If (i) is true for C_1, R_1, $C_2 R_2$

then it is also true for C. R

where $C = C_1 + C_2$

$R = R_1 \cup R_2$ ($\cup$ common bdry of R_1, R_2)





It suffices to verify R of the form