<u>Thm</u> If f is differentiable in D

and C is any curve in D
from point A to point B.
then $\int_C \underbrace{\nabla f \cdot \vec{T}\, ds}_{\nabla f \cdot d\vec{r}} = f(B) - f(A)$

<u>Pf</u>: Let $\vec{r}(t)$, $0 \le t \le 1$ be
a parametrization of C,
with $\vec{r}(0) = A$, $\vec{r}(1) = B$.
Then $\int_C \nabla f \cdot \vec{T}\, ds = \int_{t=0}^{1} \nabla f(\vec{r}(t)) \cdot \underbrace{\frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}}_{=\vec{T}} \underbrace{\left|\frac{d\vec{r}}{dt}\right| dt}_{ds}$

$= \int_0^1 \nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}(t)}{dt}\, dt = \int_0^1 \frac{d}{dt} f(\underbrace{\vec{r}(t)}_{\substack{\shortparallel \\ (x(t),\, y(t),\, z(t))}}) \cdot dt$

$\overset{\text{F.T.C.}}{=} f(\vec{r}(1)) - f(\vec{r}(0))$

Rm $\vec{F}$ is conservative in D

if for any $C \subseteq D$,

$\int_C \vec{F}\cdot\vec{T}\,ds$ only depends

on the end points of C

(independent of the path C)

The previous Theorem says.

If f is differentiable

$\Rightarrow$ ∇f is conservative.

" $\vec{F} = \nabla f \Rightarrow \vec{F}$ is conservative"

Next theorem: "$\Leftarrow$"

Thm Let D be an open connected domain, then

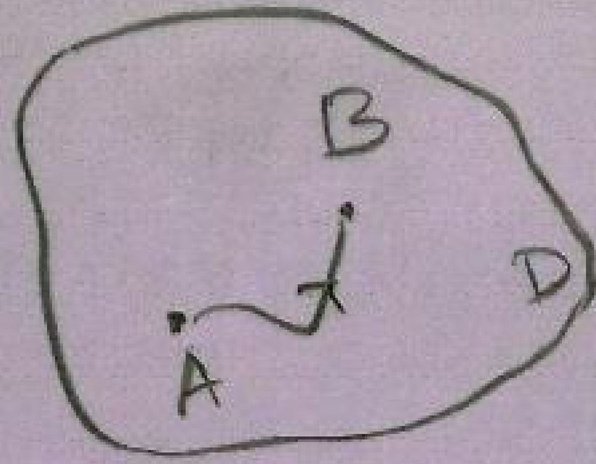
"$\vec{F}$ is conservative in D"

$\Leftrightarrow$ "$\vec{F} = \nabla f$ for some diff. function f in D"

(f = potential function of $\vec{F}$)

Pf: "$\Leftarrow$" = prev. Thm

"$\Rightarrow$": We construct f as follows:



Take any point $A \in D$.

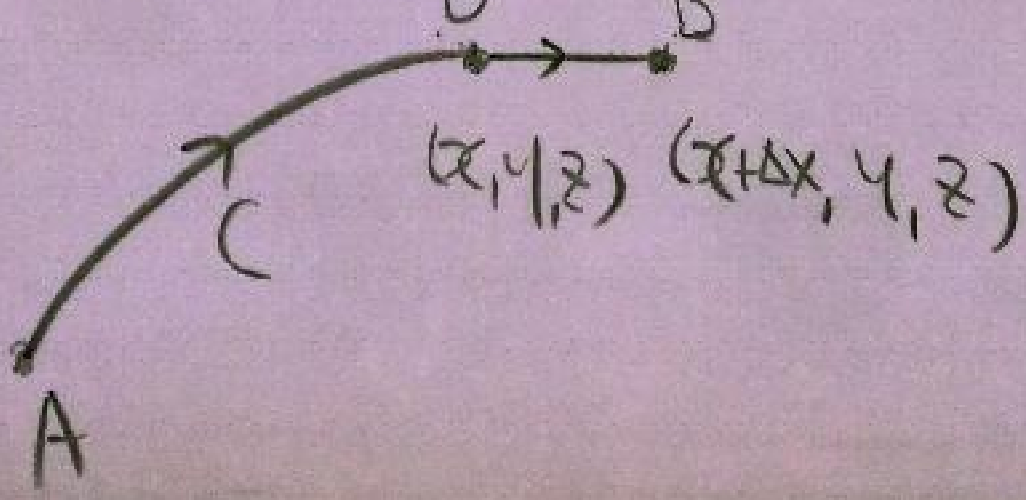
Let $f(A) = 0$

For any point B in D.

define $\underline{f(B)} = \int_C \vec{F} \cdot \vec{T}\, ds$

where C is any curve from A to B. Why $\vec{F} = \nabla f$?

Compute $f_x(x, y, z)$



$$f_x(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\int_{A \to B \to B'} \vec{F}\cdot\vec{T}\, ds - \int_{A \to B} \vec{F}\cdot\vec{T}\, ds \right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{B \to B'} \vec{F}\cdot\vec{T}\, ds$$

$$\vec{r}(t) = (t, y, z), \quad x \le t \le x+\Delta x, \quad \vec{T} = (1,0,0)$$

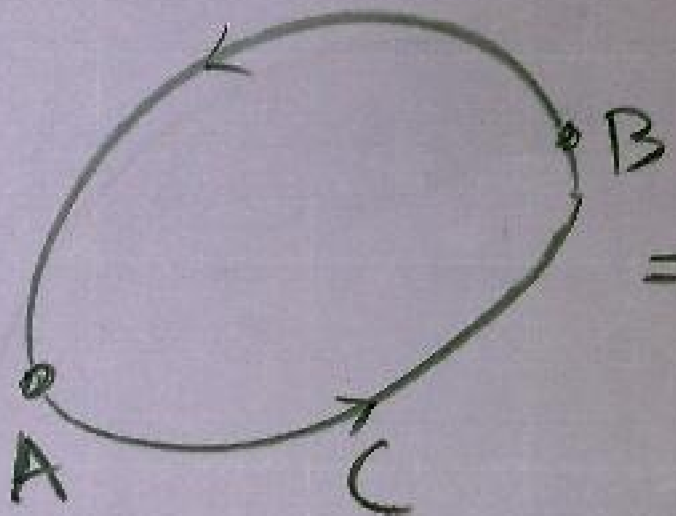
$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{t=x}^{x+\Delta x} M(t,y,z)\, dt \qquad (\vec{F} = (M, N, P))$$

$$= M(x,y,z)$$

Similarly $\begin{cases} f_y(x,y,z) = N(x,y,z) \\ f_z(x,y,z) = P(x,y,z) \end{cases}$

Rm $\vec{F}$ is conservative

$\Leftrightarrow \int_C \vec{F}\cdot\vec{T}\,ds = 0$ for any closed curve C



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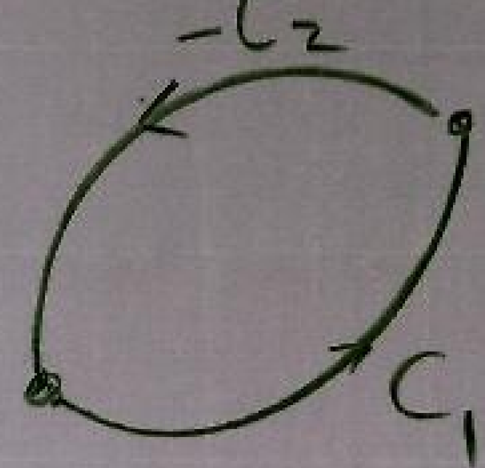

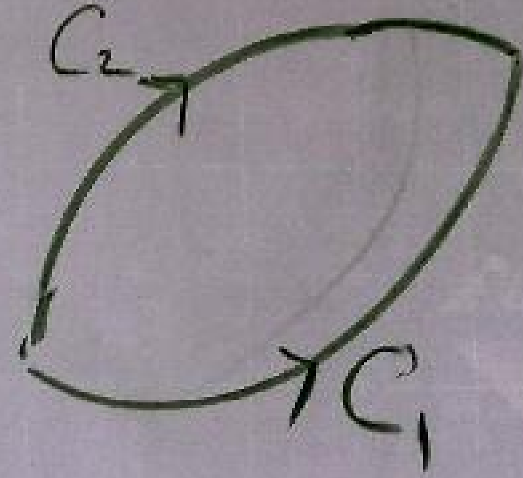

$$\int_C = \int_{C_1} + \int_{-C_2}$$

$$= \int_{C_1} - \int_{C_2} = 0$$

Remark:

(1) If $\vec{F}(x,y) = (M(x,y), N(x,y))$ is conservative.

$(M = f_x,\ N = f_y)$

$\Rightarrow M_y = N_x \ (= f_{xy})$

(necessary condition)

(2) If $\vec{F}(x,y,z)$

$= (M(x,y,z), N(x,y,z), P(x,y,z))$

is conservative, then

$$\Rightarrow \begin{cases} M_y = N_x \\ N_z = P_y \\ P_x = M_z \end{cases} \quad (*)$$

(necessary condition)

Example Show that

$$\vec{F} = (e^x \cos y + yz,\ xz - e^x \sin y,\ xy + z)$$

is conservative by finding the potential function $f(x, y, z)$

Sol (1) Check (*)

$$M_y = -e^x \sin y + z = N_x$$

$$N_z = x \qquad\qquad = P_y$$

$$P_x = y \qquad\qquad = M_z$$

((*) is satiefied)

$\Rightarrow f$ may probably exist

How to find f ?

If f exists

$$f_x = e^x \cos y + yz$$
$$f_y = xz - e^x \sin y \Rightarrow \begin{cases} f = e^x \cos y + xyz + g_1(y,z) \\ f = xyz + e^x \cos y + g_2(x,z) \\ f = xyz + \frac{z^2}{2} + g_3(x,y) \end{cases}$$
$$f_z = xy + z$$

$\Rightarrow$ try $f(x,y,z) = xyz + e^x \cos y + \frac{z^2}{2}$

check: $f_x = yz + e^x \cos y$

$f_y = xz - e^x \sin y \quad \therefore \nabla f = \vec{F}$

$f_z = xy + z$