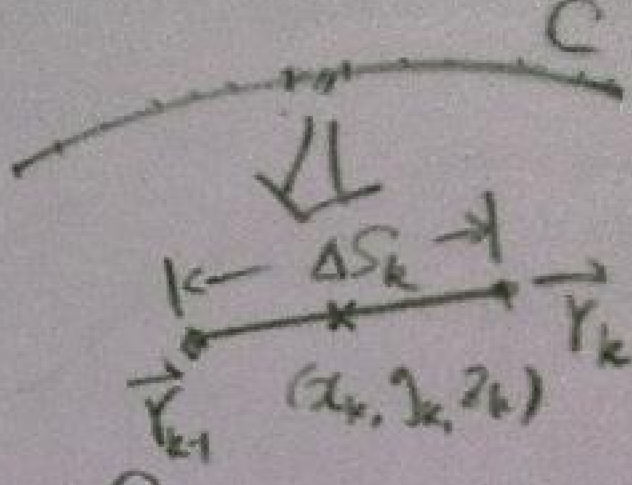
$\underline{\underline{\text{Goal}}}$: $\displaystyle\int_A^B \nabla f(\vec{r}) \cdot \vec{T}\, ds = f(B) - f(A)$




Related integrals

$\underline{(1)}$ $\displaystyle\int_C f(x, y, z)\, ds = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k) \cdot \underline{\Delta S_k}$

$$\Delta S_k = |\vec{r}_k - \vec{r}_{k-1}|$$

$f : D \longrightarrow \mathbb{R}$

$(x, y, z) \longmapsto f$

$C \subseteq D$: smooth curve

If we paramotrize $C$ as $\vec{r}(t)$ $t_0 \le t \le t_1$

$\vec{r}_k = \vec{r}(t_k)$, $\Delta S_k = \dfrac{|\vec{r}_k - \vec{r}_{k-1}|}{\Delta t_k} \Delta t_k$

$$\therefore\ dS = \left|\frac{d\vec{r}}{dt}\right| dt.$$

Example $f = x - 3y^2 + z$

C: line segment from $(0,0,0)$ to $(1,1,1)$

$$\int_C f(x,y,z)\,ds = ?$$

Sol $\vec{r}(t) = (t, t, t)$

$0 \le t \le 1$.

$$\frac{d\vec{r}}{dt} = (1,1,1), \left|\frac{d\vec{r}}{dt}\right| = \sqrt{3}$$

$$\text{Ans} = \int_{t=0}^{1} (t - 3t^2 + t)\underbrace{\sqrt{3}\,dt}_{ds} = 0$$

Example $f = x - 3y^2 + z$

$C = C_1 \cup C_2$

$C_1$ : line segment $(0,0,0) \to (1,1,0)$

$C_2$ : line segment $(1,1,0) \to (1,1,1)$

Sol $\int_C = \int_{C_1} + \int_{C_2}$

$C_1 : \vec{r_1}(t) = (t, t, 0) \quad 0 \le t \le 1$

$C_2 : \vec{r_2}(t) = (1, 1, t), \quad 0 \le t \le 1$

$$\int_{C_1} f\,ds = \int_0^1 (t - 3t^2) \underbrace{\sqrt{2}}_{\left|\frac{d\vec{r_1}}{dt}\right|} dt = \frac{-\sqrt{2}}{2}$$

$$\int_{C_2} f\,ds = \int_0^1 (-2 + t) \underbrace{dt}_{\left|\frac{d\vec{r_2}}{dt}\right|} = \frac{-3}{2}$$

Related integral

(2) $\int_C \vec{F} \cdot \vec{T} \, ds$

$\vec{F} : D \to \mathbb{R}^3$

$(x, y, z) \qquad (F_1, F_2, F_3)$

$C$: a smooth curve with prescribed orientation

$\vec{r}(t)$: parametrization of $C$ where "direction of increasing $t$" = "orientation of $C$"

$$\vec{T} = \frac{d\vec{r}(t)}{dt} \Big/ \left| \frac{d\vec{r}(t)}{dt} \right|$$

where $\vec{r}(t)$ is as above.

Example: $\vec{F} = (z, xy, -y^2)$

$C: \vec{r}(t) = (t^2, t, \sqrt{t}), 0 \le t \le 1$

$$\frac{d\vec{r}(t)}{dt} = \left(2t, 1, \frac{1}{2\sqrt{t}}\right)$$

$$\int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_C \vec{F}(\vec{r}(t)) \cdot \overbrace{\left(\frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}\right)}^{\vec{T}} \underbrace{\left|\frac{d\vec{r}}{dt}\right| dt}_{ds}$$

$$= \int_C \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt \quad \left( = \int_C \vec{F} \cdot d\vec{r} \right)$$

Here $\vec{F}(\vec{r}(t)) = \vec{F}(t^2, t, \sqrt{t}) = (\sqrt{t}, t^3, -t^2)$

<u>Ans</u>: $\int_0^1 (\sqrt{t}, t^3, -t^2) \cdot \left(2t, 1, \frac{1}{2\sqrt{t}}\right) dt$

<u>Remark</u>: we sometimes write

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

But we still need explicit parametrization $x(t), y(t), z(t)$

to compute it as

$$\int_{t_1}^{t_2} \begin{pmatrix} F_1(x(t), y(t), z(t))\,\frac{dx}{dt} \\ + F_2(x(t), y(t), z(t))\,\frac{dy}{dt} \\ + F_3(x(t), y(t), z(t))\,\frac{dz}{dt} \end{pmatrix} dt$$

<u>Remark</u>: If $\vec{F}$ = force,

then $\int_C \vec{F}\cdot\vec{T}\,ds$ = work done

Remark: If $C$ is a simple closed curve, we use the notation

$$\oint_C \vec{F}\cdot\vec{T}\,ds$$ to specify

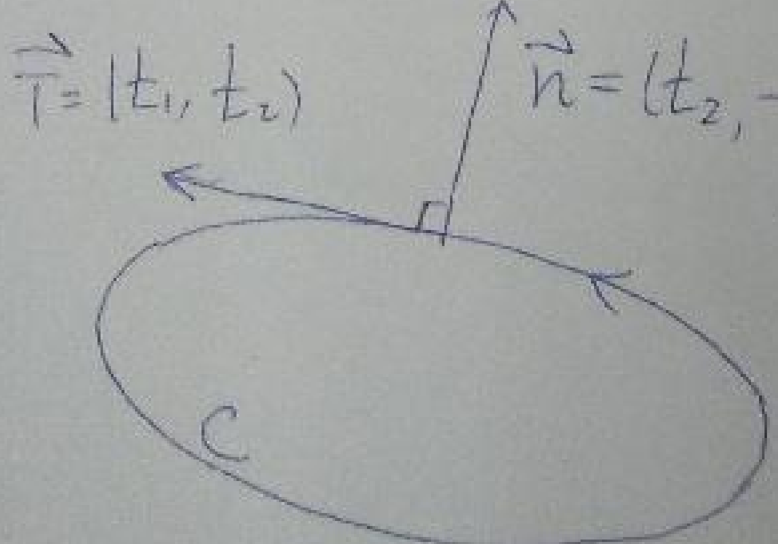
the orientation of $C$ (counter-clockwise)

Remark: If $C$ is a simple closed curve

$\vec{r}(t)$ is a counter clockwise paramotrization of $C$, then $\vec{T} = \frac{d\vec{r}(t)}{dt} \Big/ \left|\frac{d\vec{r}(t)}{dt}\right|$,

is the unit tangent vector pointing counter clockwise.

Let $\vec{n}$ = outward normal (See below)




We have

$$\boxed{\begin{aligned}&\vec{T} = (t_1, t_2)\\ \Rightarrow\ &\vec{n} = (t_2, -t_1)\end{aligned}}$$

Related integral

(3) $\int_C \vec{F} \cdot \vec{n} \, ds$ (flux)

C: simple closed curve.

$\vec{n}$: outward normal.

Standard approach:

$\vec{r}(t)$: counter clockwise

$$\vec{T} = \frac{d\vec{r}(t)}{dt} \Big/ \left| \frac{d\vec{r}(t)}{dt} \right|$$

$$\frac{d\vec{r}(t)}{dt} = \left( \frac{dx(t)}{dt}, \frac{dy(t)}{dt} \right)$$

$$\vec{n} = \left( \frac{dy(t)}{dt}, -\frac{dx(t)}{dt} \right) \Big/ \left| \frac{d\vec{r}(t)}{dt} \right|$$

Example $\vec{F} = (x-y, y)$

$C: x^2+y^2=1$, $\int_C \vec{F}\cdot\vec{n}\, ds = ?$

Sol: Counter-clock wise parametrization $0 \leq t \leq 2\pi$

$\vec{r}(t): x(t) = \cos t,\ y(t) = \sin t$

$$\vec{T} = \frac{(x'(t), y'(t))}{\left|\frac{d\vec{r}}{dt}\right|} = (-\sin t, \cos t)$$

$$\vec{n} = (y'(t), -x'(t)) \cancel{/\left|\frac{d\vec{r}}{dt}\right|} = (\cos t, \sin t)$$

Ans: $$\int_0^{2\pi} (F_1, F_2) \cdot \underbrace{\frac{\left(\frac{dy}{dt}, \frac{-dx}{dt}\right)}{\left|\frac{d\vec{r}}{dt}\right|}}_{\vec{n}} \underbrace{\left|\frac{d\vec{r}}{dt}\right| dt}_{ds}$$

$$\left( \begin{aligned} &= \oint_C F_1\,dy - F_2\,dx = \int_0^{2\pi} \left(F_1 \frac{dy}{dt} - F_2 \frac{dx}{dt}\right) dt \\ &= \oint_C M\,dy - N\,dx = \int_0^{2\pi} \left(M \frac{dy}{dt} - N \frac{dx}{dt}\right) dt \\ &M = F_1,\ N = F_2 \text{ in textbook} \end{aligned} \right)$$

$$= \int_0^{2\pi} (F_1, F_2) \cdot \left(\frac{dy}{dt}, -\frac{dx}{dt}\right) dt$$

$$= \int_0^{2\pi} (\cos t - \sin t, \sin t) \cdot (\cos t, \sin t)\, dt$$

$$= \int_0^{2\pi} (\cos t - \sin t)\cos t + \sin^2 t \, dt$$