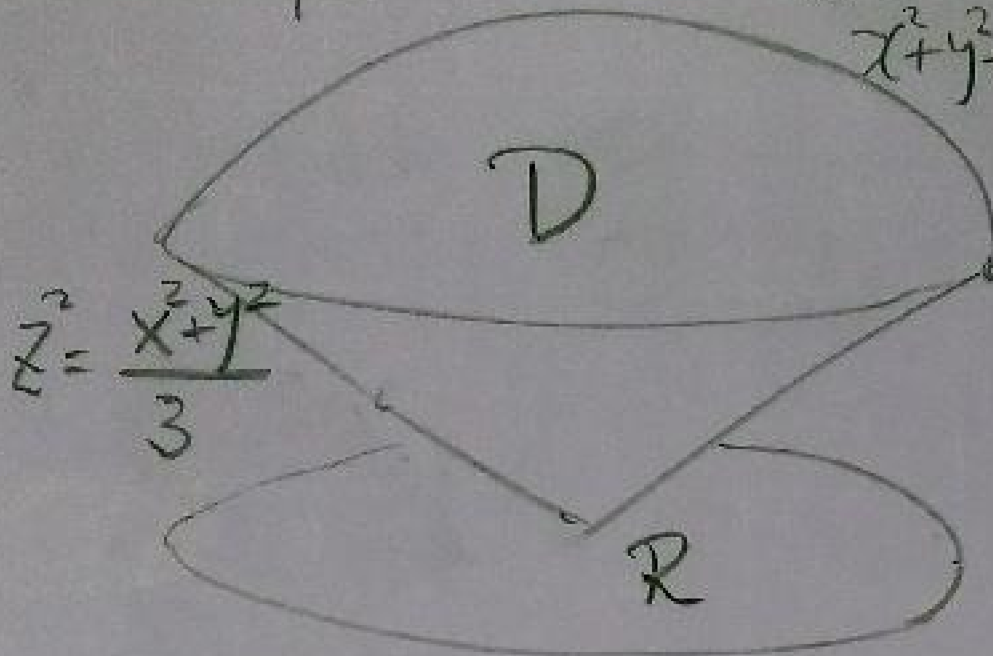
Example Find volume of the domain D given by

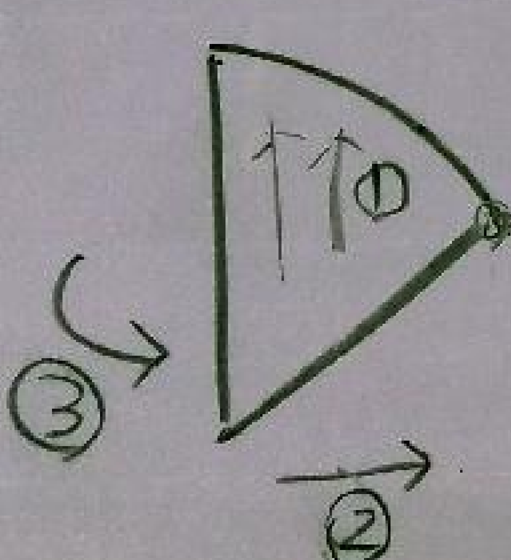


$$\begin{cases} x^2+y^2+z^2 \le 1 \\ z \ge \sqrt{\dfrac{x^2+y^2}{3}} \end{cases}$$

Sol: (I): Cylindrical Cordinate

$$R = \left\{ x^2+y^2+\frac{x^2+y^2}{3} \le 1 \right\}$$

$$= \left\{ x^2+y^2 \le \frac{3}{4} \right\}$$

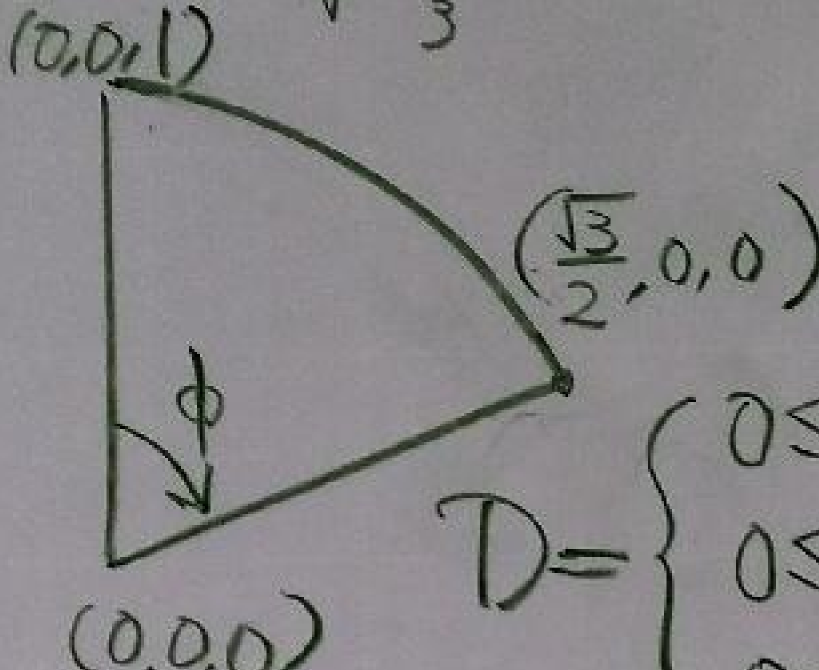


$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{\frac{\sqrt{3}}{2}} \int_{z=\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} 1 \, dz \, r \, dr \, d\theta$$

(II) Spherical Coordinate

$$x^2+y^2+z^2 \le 1 \iff \rho \le 1$$

$$z \ge \sqrt{\frac{x^2+y^2}{3}} \iff 0 \le \phi \le \frac{\pi}{3}$$



$$D = \begin{cases} 0 \le \rho \le 1 \\ 0 \le \phi \le \frac{\pi}{3} \\ 0 \le \theta \le 2\pi \end{cases}$$

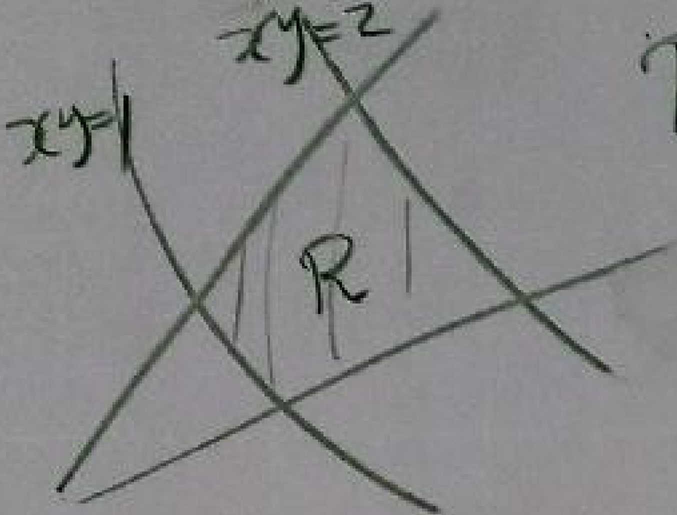
$$D = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^1 \rho^2 d\rho \int_0^{\frac{\pi}{3}} \sin\phi \, d\phi \int_0^{2\pi} d\theta$$

$$= \frac{1}{3} \cdot \frac{1}{2} \cdot 2\pi = \frac{\pi}{3}$$

Substitution in Multiple integrals

Example Find area of



$$R = \left\{ \begin{array}{l} 1 \le xy \le 2 \\ 1 \le \frac{y}{x} \le 2 \end{array} \right\}$$

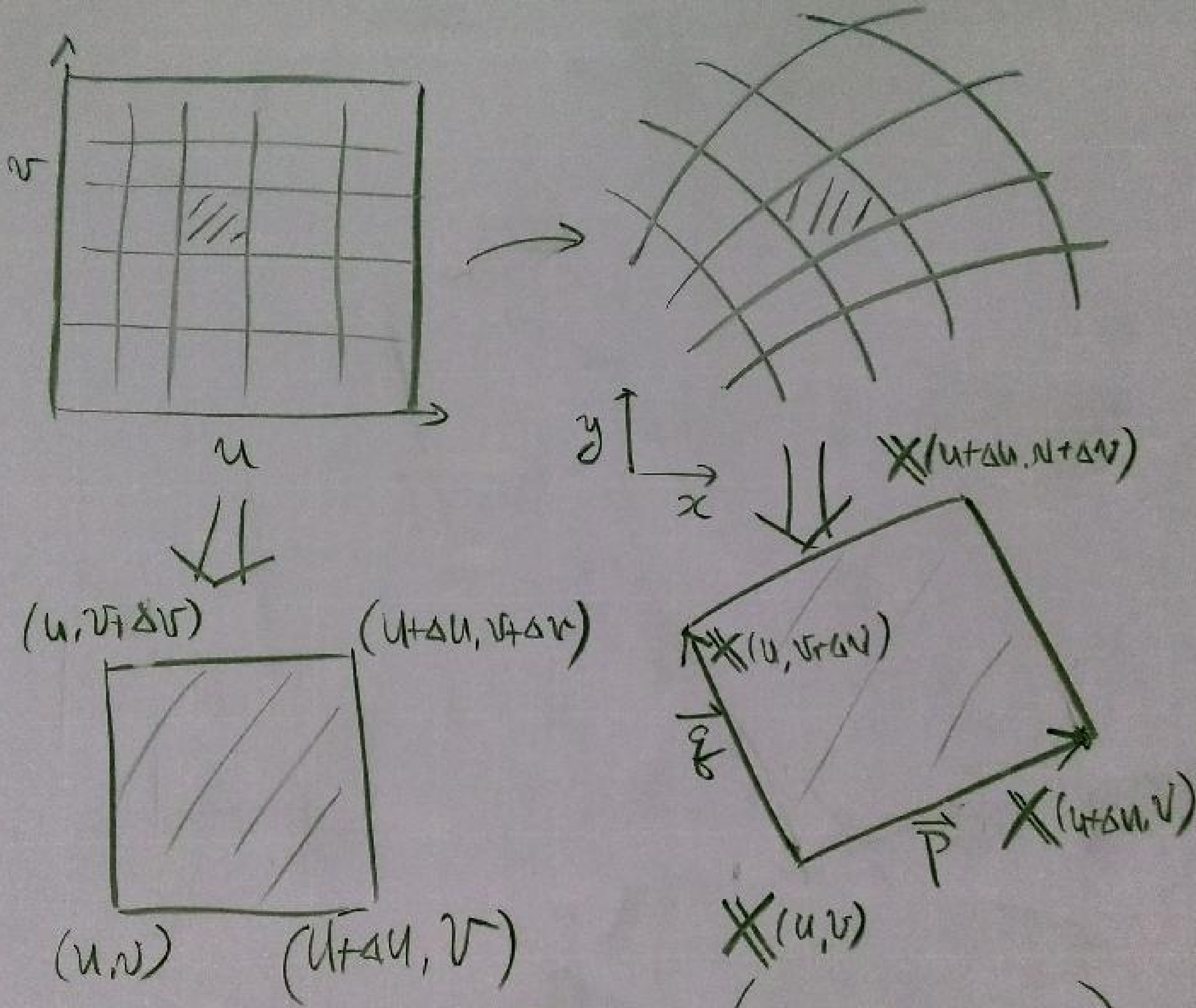
in first quadrant.

Sol Let $u = xy$. $v = \frac{y}{x}$

$$A = \iint_R dA = \int_{v=1}^{2} \int_{u=1}^{2} \underline{dA}$$

$dA = ?$ in $\underline{du}\, dv$

Here $x = \sqrt{\frac{u}{v}}$, $y = \sqrt{u.v}$.



$$\mathbb{X}(u,v) = \big(x(u,v),\ y(u,v)\big)$$

etc.

$$\Delta A$$

$$\approx \left| \left(\mathbb{X}(u+\Delta u, v) - \mathbb{X}(u,v) \right) \times \left(\mathbb{X}(u, v+\Delta v) - \mathbb{X}(u,v) \right) \right|$$

$$= \left| \vec{p} \times \vec{q} \right|$$

$$\vec{p} = \left(x(u+\Delta u, v) - x(v,v),\ y(u+\Delta u, v) - y(u,v) \right)$$

$$= \left(\frac{\partial x}{\partial u}(\circ)\,\Delta u \ ,\ \frac{\partial y}{\partial u}(*)\,\Delta u \right)$$

$$(\circ) = (u + c_1 \Delta u, v) \ ,\ (*) = (u + c_2 \Delta u, v)$$

$$\vec{q} = \left(\frac{\partial x}{\partial v}(\circ\circ)\,\Delta v,\ \frac{\partial y}{\partial v}(**)\,\Delta v \right)$$

$$\vec{p} \times \vec{q} = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}_{J} \Delta u \Delta v$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = J \qquad \Delta A = |J|\, \Delta u \Delta v$$

Back to our example

$$A = \int_1^2 \int_1^2 |J| \, du\,dv$$

$$J = \begin{vmatrix} \frac{1}{2}\frac{1}{\sqrt{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix}$$

$$= \frac{1}{4}\left(\frac{1}{v} + \frac{1}{v}\right) = \frac{1}{2v}$$

$$A = \int_1^2 \int_1^2 \left|\frac{1}{2v}\right| du\,dv = \frac{\ln 2}{2}$$

Summary:

Step 1: Find suitable $u(x,y)$, $v(x,y)$ and Solve for $x(u,v)$, $y(u,v)$

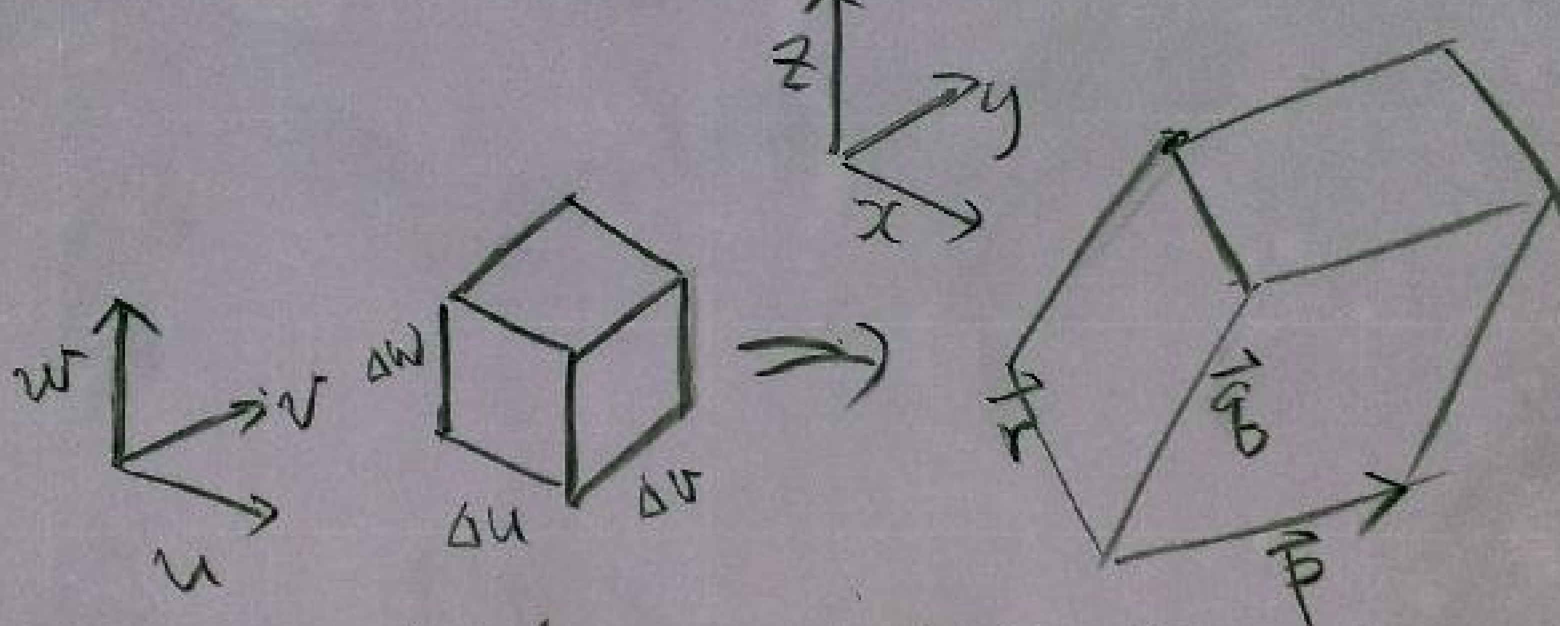
Step 2 Compute $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

Step 3 $dA = |J| \, du\,dv$

For triple integrals

Using $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$ as integration variables

$$\Delta V \stackrel{?}{=} \text{in}\,(\Delta u, \Delta v, \Delta w)$$



$$\Delta V = |J|, \quad J = \vec{p} \times \vec{q} \cdot \vec{r}$$

$$\mathbb{X} = (x, y, z) \qquad = \left| \vec{p} \;\; \vec{q} \;\; \vec{r} \right|$$

$$\vec{p} = \mathbb{X}(u+\Delta u, v, w) - \mathbb{X}(u, v, w) \cong \mathbb{X}_u \, \Delta u$$

$$\vec{q} = \mathbb{X}(u, v+\Delta v, w) - \mathbb{X}(u, v, w) \cong \mathbb{X}_v \, \Delta v$$

$$\vec{r} = \mathbb{X}(u, v, w+\Delta w) - \mathbb{X}(u, v, w) \cong \mathbb{X}_w \, \Delta w$$

$$J = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$\Delta V = |J| \, \Delta u \, \Delta v \, \Delta w$$

__Step 1__ $u(x,y,z), v(x,y,z), w(x,y,z)$

$\Longrightarrow x(u,v,w), y(u,v,w), z(u,v,w)$

__Step 2__ $J = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$

__Step 3__ $dx\,dy\,dz = |J| \, du\,dv\,dw$