Triple integrals in Cylindrical and Spherical coordinates.

I. Cylindrical coordinate

= polar coordinate + z coordinate

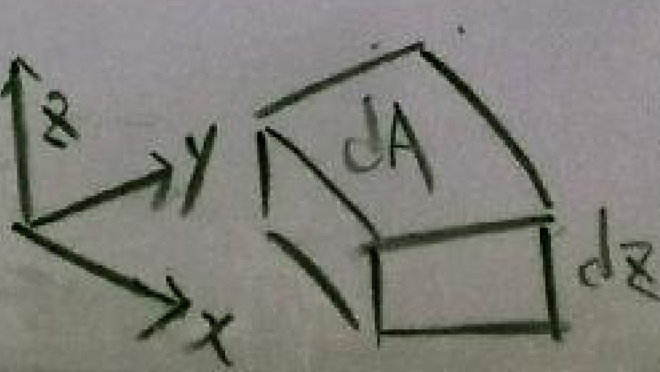
$$(x, y, z) \longleftrightarrow (r, \theta, z)$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}, \quad z = z$$

$$\theta = \begin{cases} \tan^{-1}\frac{y}{x} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), & (x,y) \in \frac{I}{IV} \\ \tan^{-1}\frac{y}{x} \pm \pi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \; \left(\frac{-3\pi}{2}, \frac{\pi}{2}\right), & (x,y) \in \frac{II}{III} \end{cases}$$

$$dV = dA \cdot dz$$
$$= r\,dr\,d\theta\,dz$$

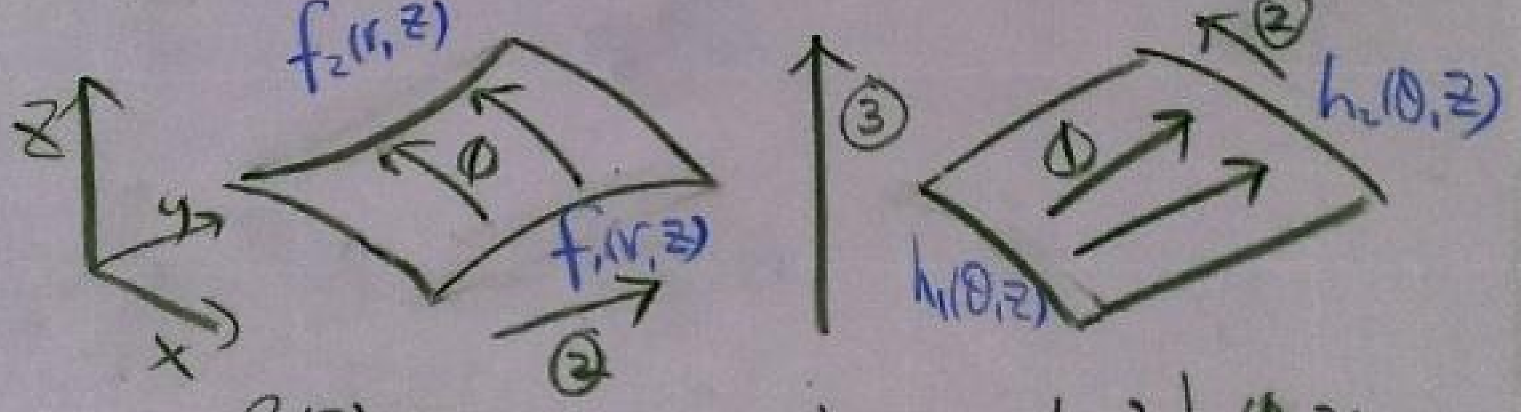


It remains to find limits of integrations for different ordering of $r\,dr\,d\theta\,dz$, $d\theta\,dz\,r\,dr$, etc.

Case I $d\theta\,r\,dr\,dz$ or $r\,dr\,d\theta\,dz$

3rd variable is "z"

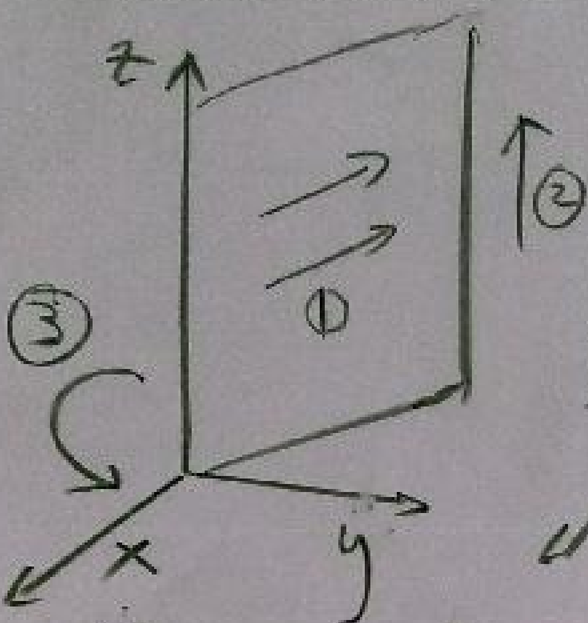
Start with cross sections "z = const"



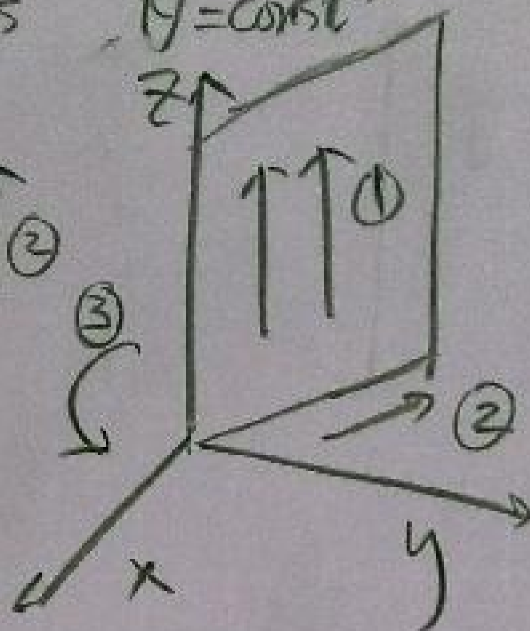
$$\int_{z_1}^{z_2}\int_{g_1(z)}^{g_2(z)}\int_{f_1(r,z)}^{f_2(r,z)} (*)\, d\theta\, r\,dr\,dz \qquad \Bigg| \qquad \int_{z_1}^{z_2}\int_{k_1(r)}^{k_2(r)}\int_{h_1(\theta,z)}^{h_2(\theta,z)} (*)\, r\,dr\,d\theta\,dz$$

Case II $r\,dr\,dz\,d\theta$ or $dz\,r\,dr\,d\theta$

Cross sections "$\theta = $ const"



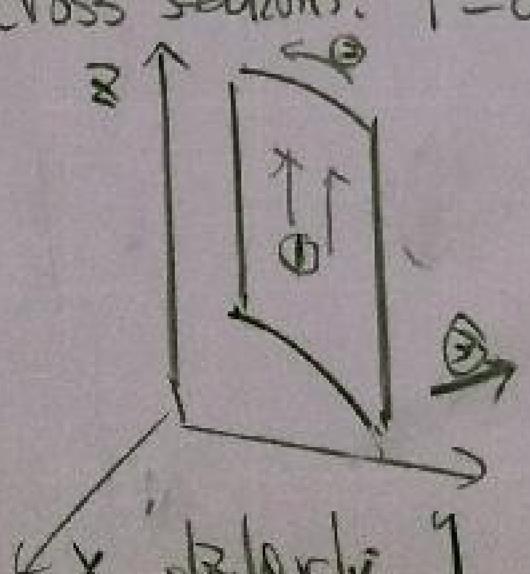
"$r\,dr\,dz\,d\theta$"



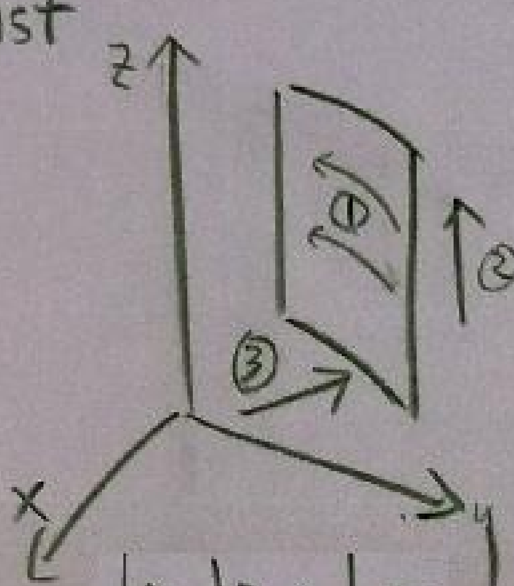
"$dz\,r\,dr\,d\theta$"

Case III $dz\,d\theta\,r\,dr$ or $d\theta\,dz\,r\,dr$

Cross sections: "$r = $ const"

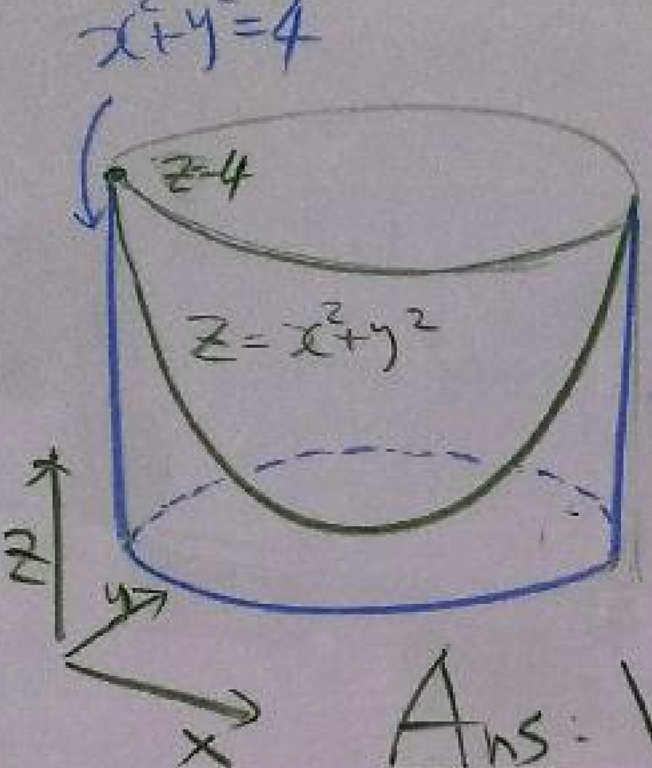


$dz\,d\theta\,r\,dr$



$d\theta\,dz\,r\,dr$

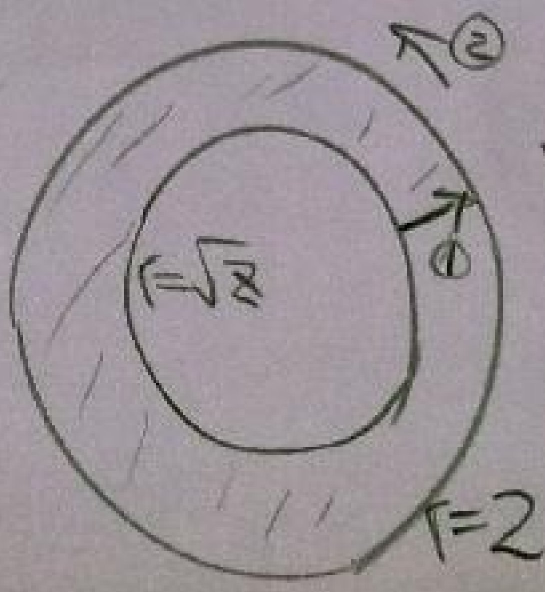
Example $D=\left\{ \begin{array}{l} x^2+y^2\le 4 \\ 0\le z\le x^2+y^2 \end{array} \right\}$



Find volume of D using Cylindrical coordinate:

Ans: $V=\iiint_D 1\,dV$

Case I $r\,dr\,d\theta\,dz$

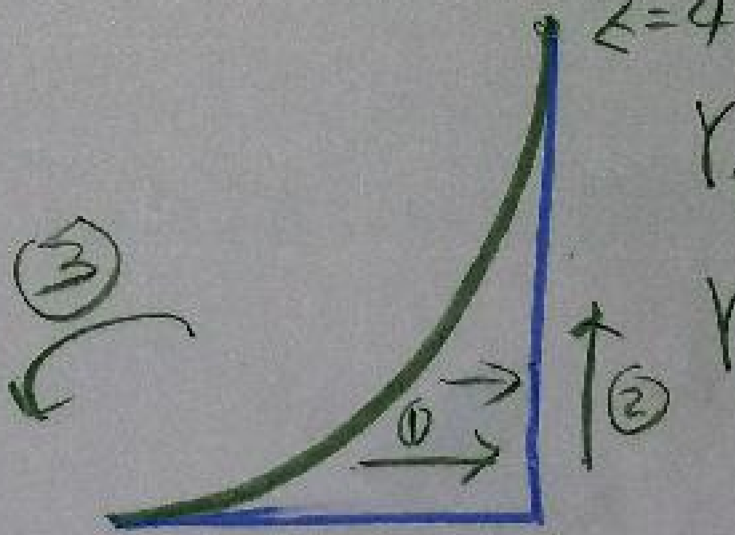


$$V=\int_0^4\int_0^{2\pi}\int_{\sqrt{z}}^{2} 1\,r\,dr\,d\theta\,dz$$

Here $z_1=0$ and

$z_2: \{x^2+y^2=4\}\cap\{z=x^2+y^2\}$

Case II $r\,dr\,dz\,d\theta$, $dz\,r\,dr\,d\theta$



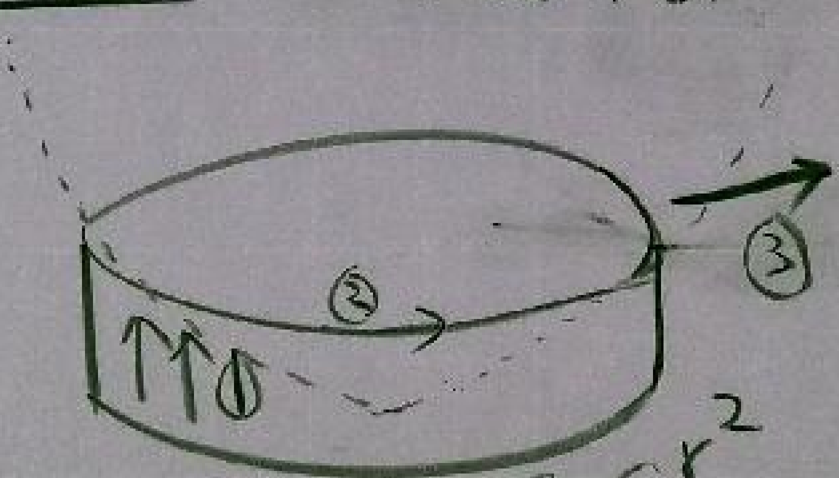
$r_{min}: z=r^2$

$r_{max}: r=2$

$$V=\int_0^{2\pi}\int_0^{4}\int_{\sqrt{z}}^{2} 1\, r\,dr\,dz\,d\theta$$

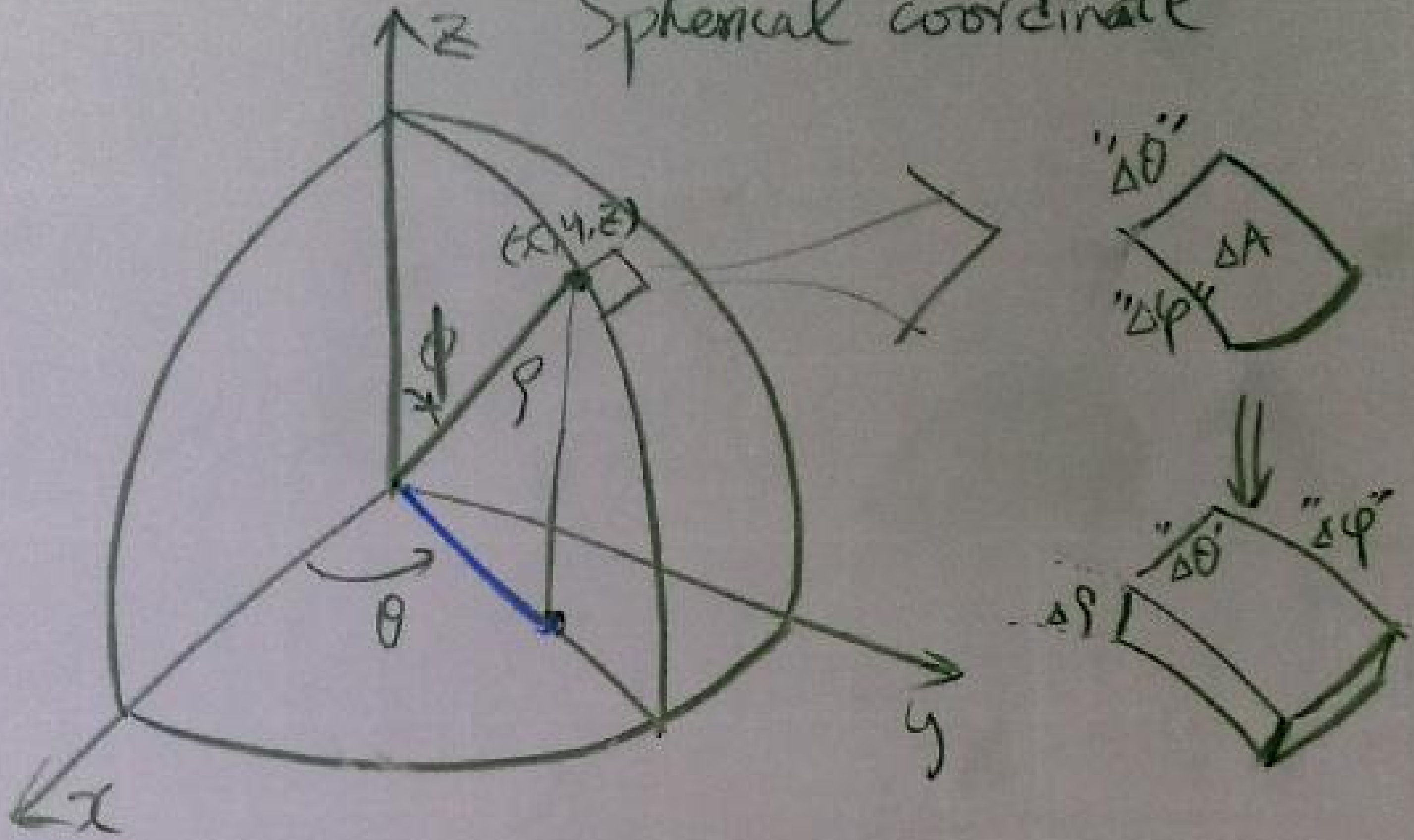
$$=\int_0^{2\pi}\int_0^{2}\int_0^{r^2} dz\, r\,dr\,d\theta$$

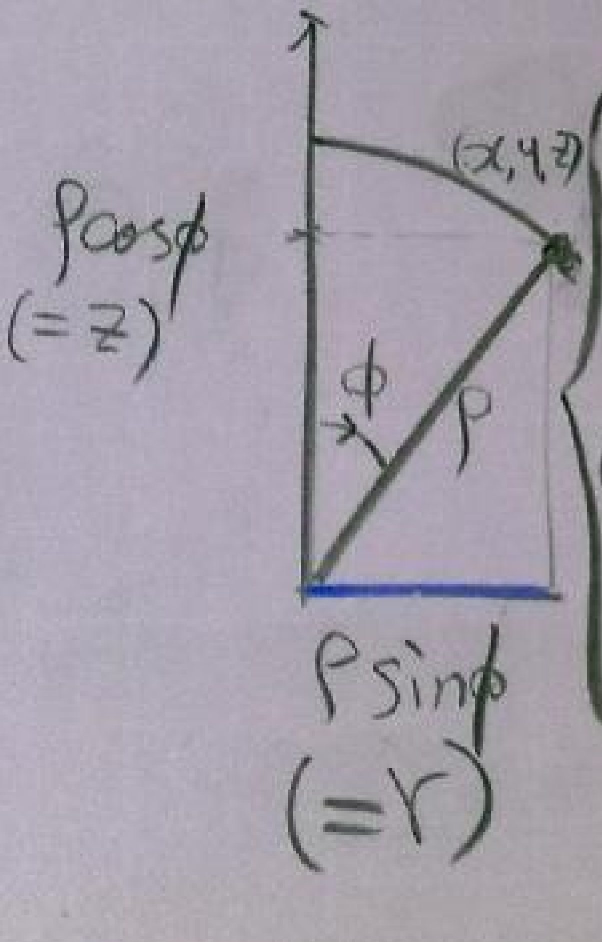
Case III $dz\,d\theta\,r\,dr$



$$V=\int_0^{2}\int_0^{2\pi}\int_0^{r^2} dz\,d\theta\,r\,dr$$

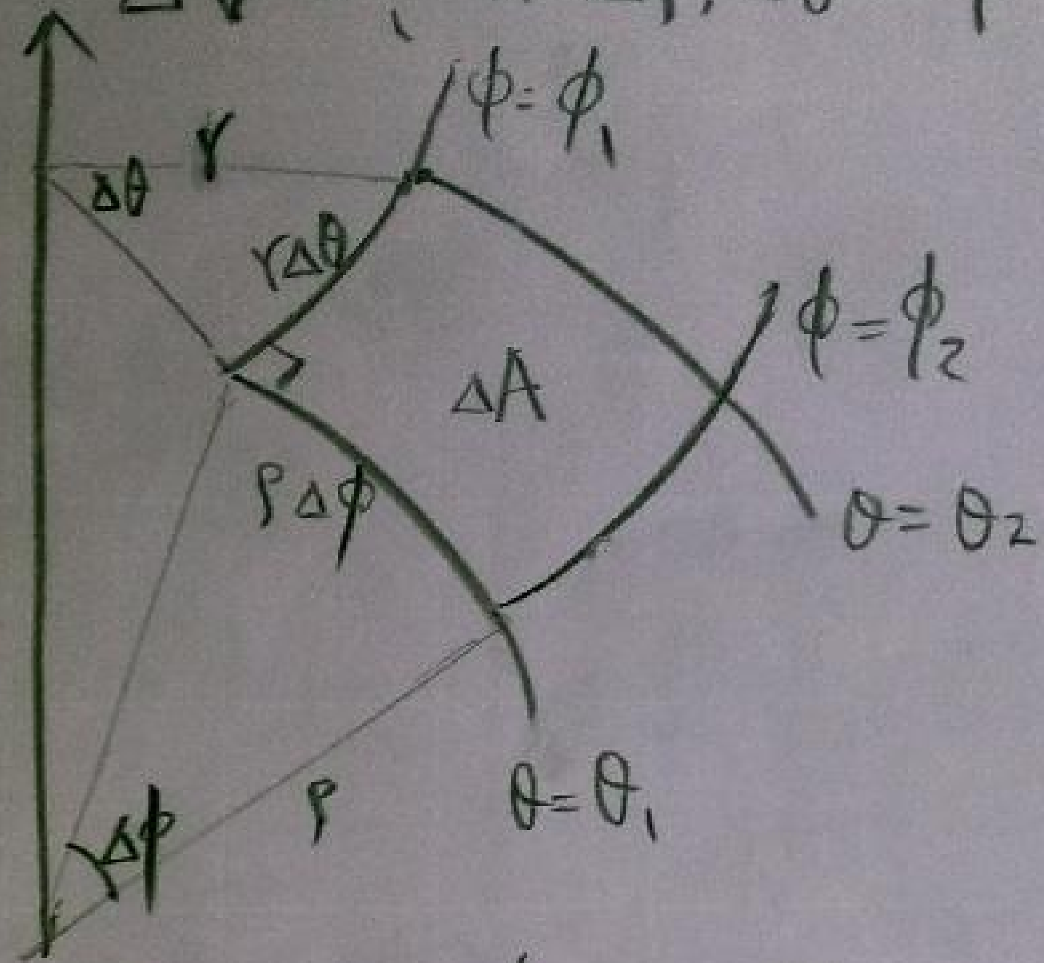
Spherical coordinate

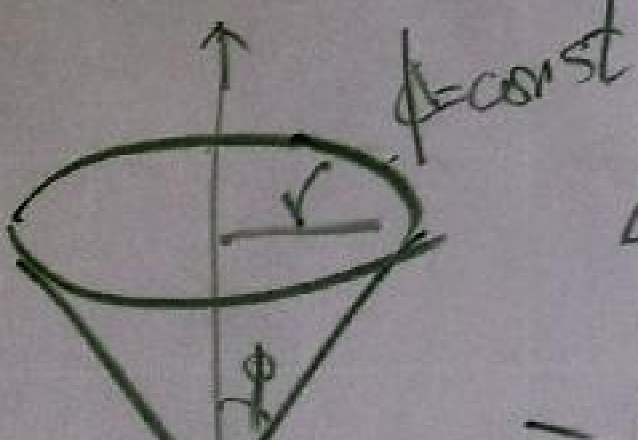




$$\begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \cot\phi = \dfrac{z}{\sqrt{x^2+y^2}} \\ \left(\cos\phi = \dfrac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \tan\theta = \dfrac{y}{x} \end{cases}, \quad \begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases},$$

$\Delta V = ?$ in $\Delta\rho, \Delta\theta, \Delta\phi$



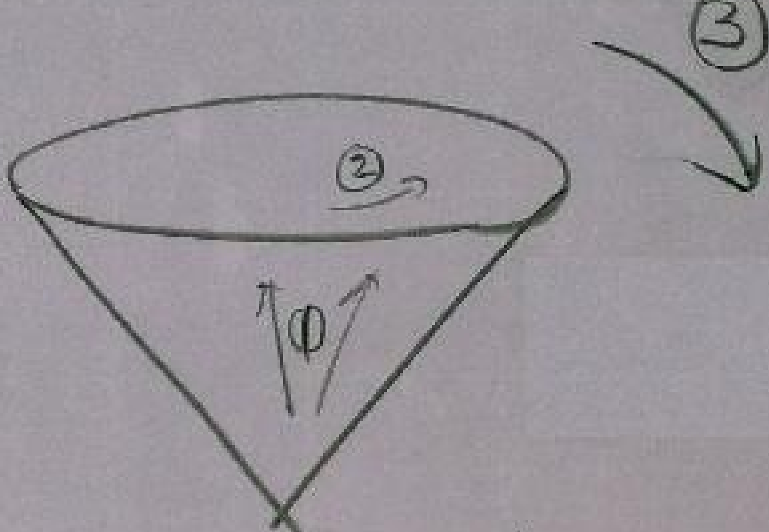


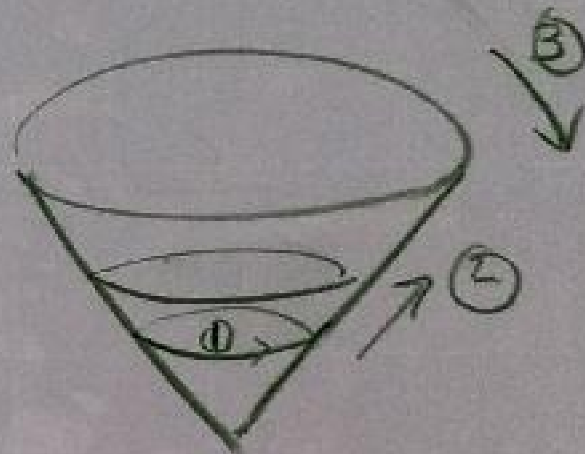
$$\Delta A = (\rho\Delta\phi)(r\Delta\theta)$$
$$= \rho\Delta\phi\,\rho\sin\phi\,\Delta\theta$$

$$\Delta V = \Delta A\,\Delta\rho$$
$$= \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

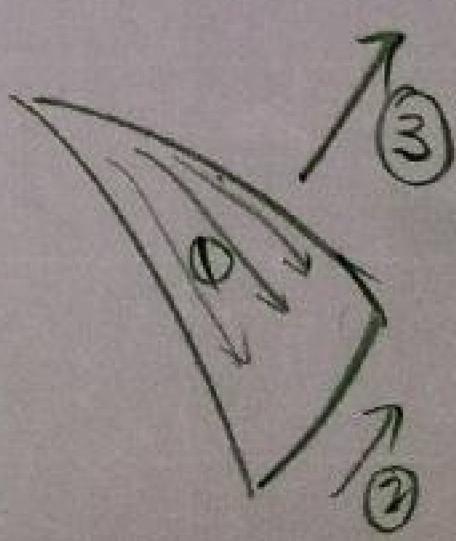
Cross sections for various order of integration

Case I $d\rho\, d\theta\, d\phi$ $d\theta\, d\rho\, d\phi$





Case II $d\theta\, d\phi\, d\rho$ $d\phi\, d\theta\, d\rho$



Case III $d\rho\, d\phi\, d\theta$, $d\phi\, d\rho\, d\theta$