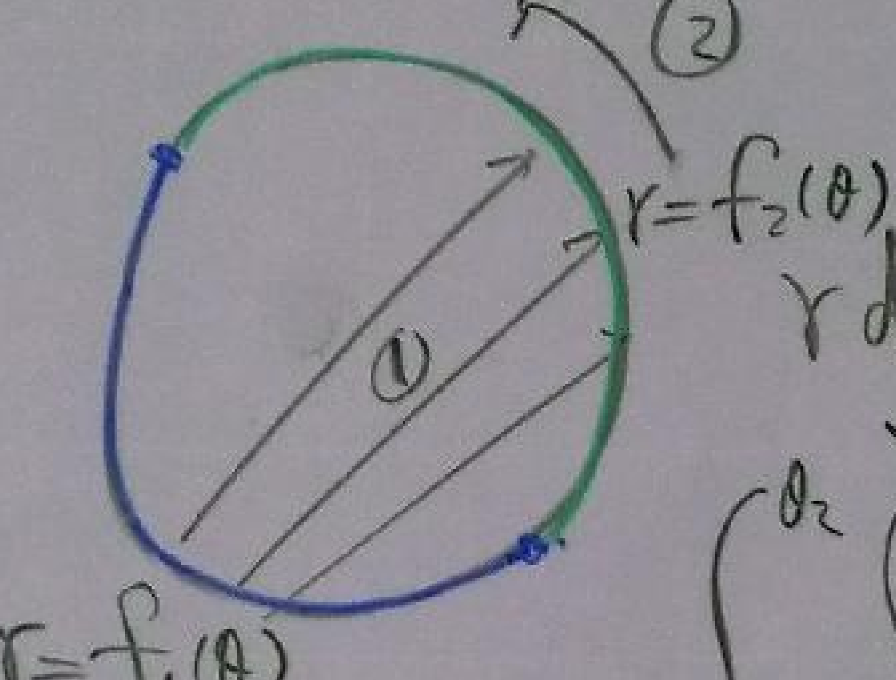
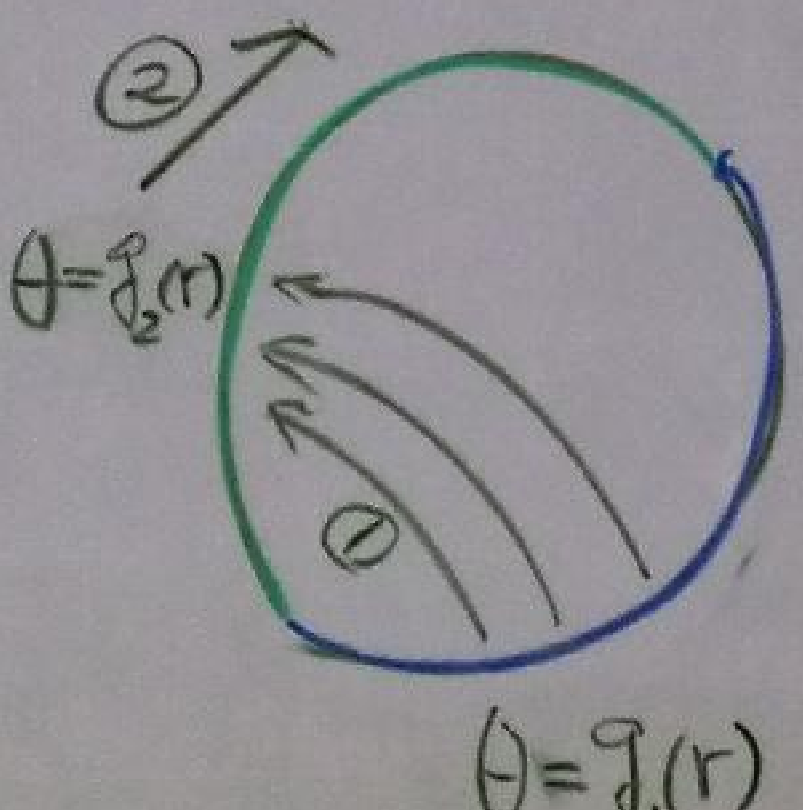
Double integral in Polar Coord.

(i) $dA = r\,dr\,d\theta$

(ii)



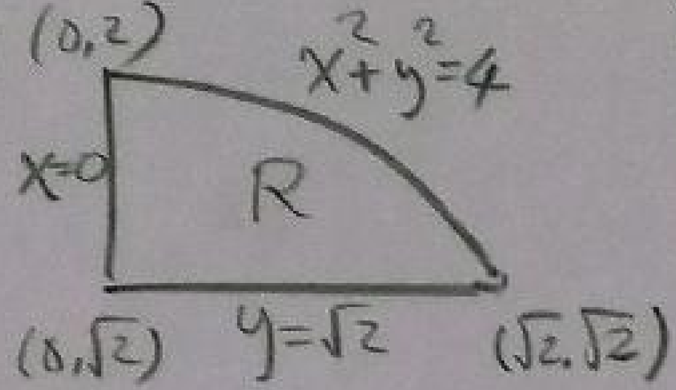
$$r\,dr\,d\theta$$
$$\Downarrow$$
$$\int_{\theta_1}^{\theta_2}\int_{f_1(\theta)}^{f_2(\theta)} * \; r\,dr\,d\theta$$



$$d\theta\; r\,dr$$
$$\Downarrow$$
$$\int_{r_1}^{r_2}\int_{g_1(r)}^{g_2(r)} * \; d\theta\; r\,dr$$

Examples

(i)

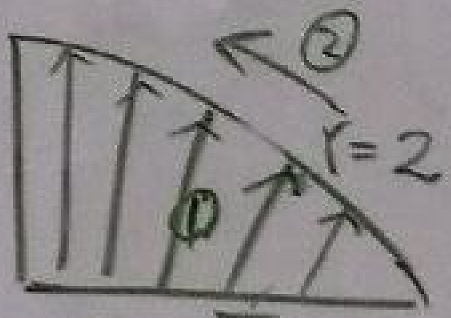


$x^2+y^2=4 \Leftrightarrow r=2$

$x=0 \Leftrightarrow r\cos\theta=0$

$y=\sqrt{2} \Leftrightarrow r\sin\theta=\sqrt{2}$

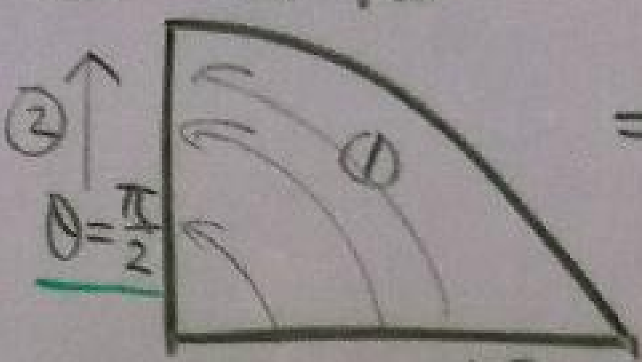
ⓐ $r\,dr\,d\theta$



$r=\frac{\sqrt{2}}{\sin\theta}$

$$\Rightarrow \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\sqrt{2}}{\sin\theta}}^{2} (*)\, r\,dr\,d\theta$$

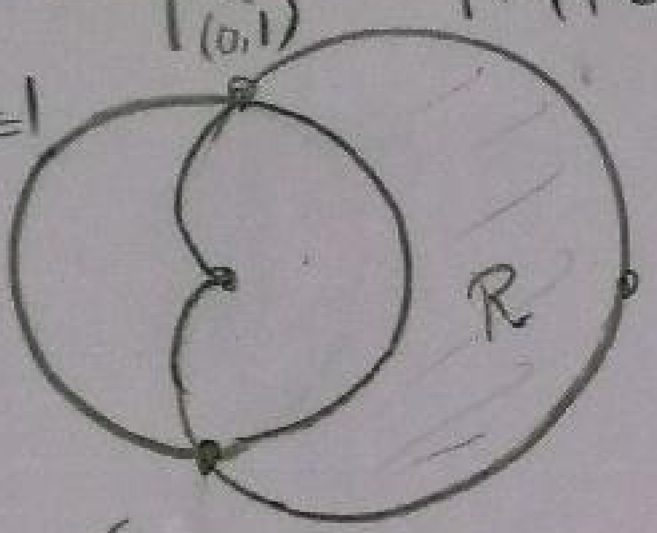
ⓑ $d\theta\, r\,dr$



$\theta=\sin^{-1}\left(\frac{\sqrt{2}}{r}\right)$

$$\Rightarrow \int_{\sqrt{2}}^{2} \int_{\sin^{-1}\left(\frac{\sqrt{2}}{r}\right)}^{\frac{\pi}{2}} (*)\, d\theta\, r\,dr$$

Example

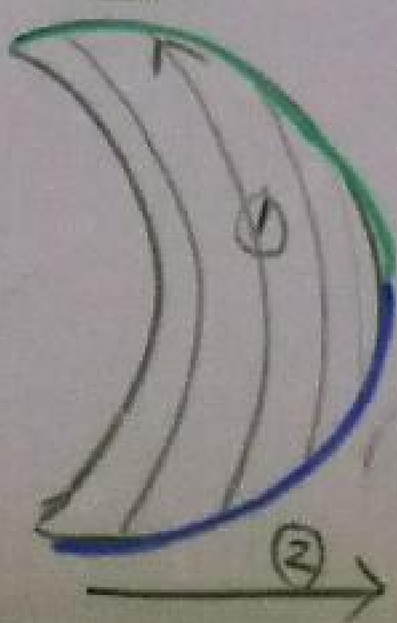


(a)

$r\,dr\,d\theta$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{1+\cos\theta} (*)\, r\,dr\,d\theta$$

(b)



$d\theta\, r\,dr \Rightarrow \int_{1}^{2} \int_{-\cos^{-1}(r-1)}^{\cos^{-1}(r-1)} (*)\, d\theta\, r\,dr$

$r = 1+\cos\theta$

$\theta = \cos^{-1}(r-1) \in [0, \frac{\pi}{2}]$

$(\cos^{-1}x \in [0,\pi])$

$\theta = -\cos^{-1}(r-1) \in [-\frac{\pi}{2}, 0]$

Remark.

$r\,dr\,d\theta$ is usually a better choice than $d\theta\, r\,dr$. since curves in polar coordinate are generally expressed as $r = f(\theta)$, which is needed in $\int_{*}^{*} r\,dr$.

Triple integrals in Cartesian Coord.

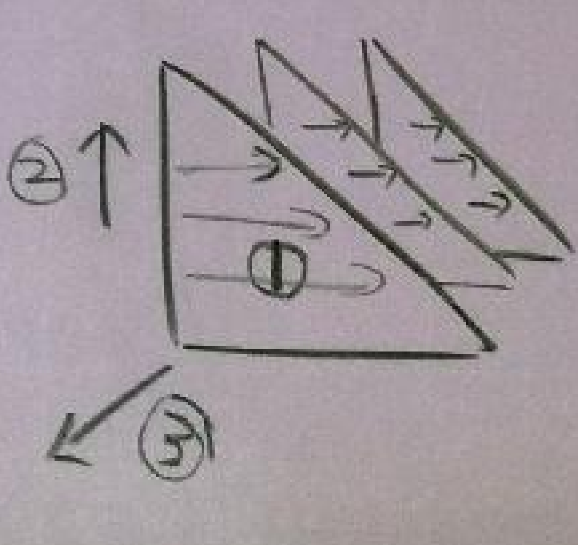
$$\iiint_D f(x,y,z)\,dV = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k,y_k,z_k)\,\Delta V_k$$

when $f(x,y,z)=1$, integral = volume of D

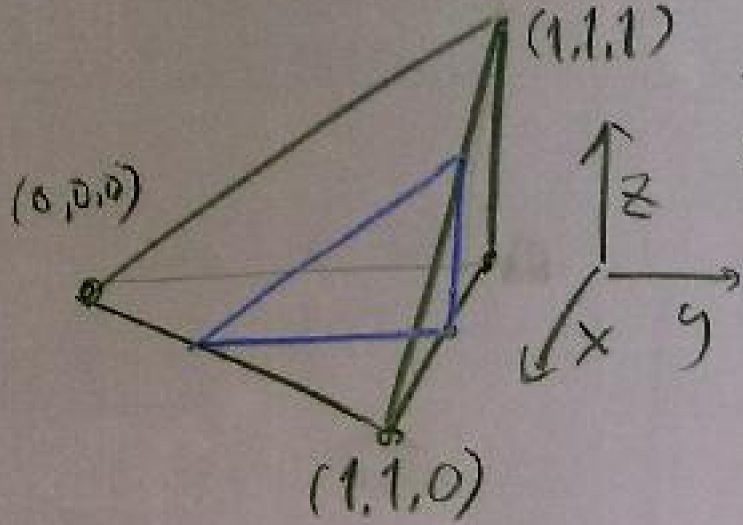
(i) In cartesian coordinate

$$dV = dx\,dy\,dz = dy\,dz\,dx = \cdots$$

(ii) $d1\;d2\;d3$



Example : Volume = ?



: $x = 0$ (A)

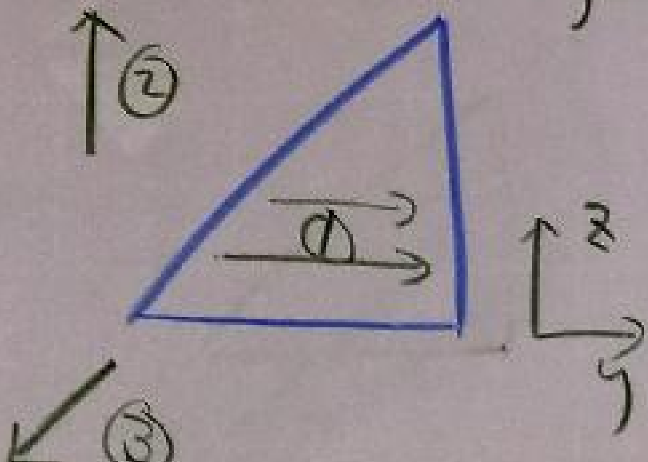
: $z = 0$ (B)

: $y = 1$ (C)

: $y = x + z$ (D)

Case ① $dy\, dz\, dx$

Step1: Sketch the cross sections in y-z directions, i.e. $x = \text{const}$



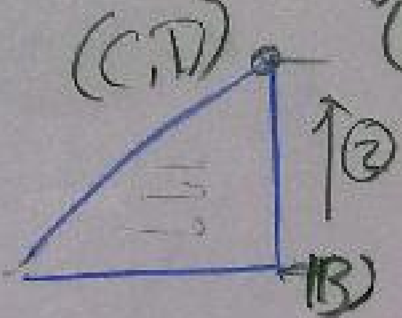
$$dy \Rightarrow \int_{(D)}^{(C)} dy = \int_{x+z}^{1} dy$$

$$\left.\begin{array}{l} (D):\ y = x + z \\ (C):\ y = 1 \end{array}\right\}$$

expressed as ① = F(②,③)

$y = F(z, x)$

Step 2 $\displaystyle\int_{(B)}^{(C,D)} \int_{x+z}^{1} dy\, dz$



$$= \int_{z=0}^{1-x} \int_{x+z}^{1} dy\, dz$$

$$\left.\begin{array}{l} (B):\ z = 0 \\ (C,D)\ \begin{cases} y = 1 \\ y = x+z \end{cases} \\ \quad \therefore z = 1-x \end{array}\right\}$$

② = G(③)

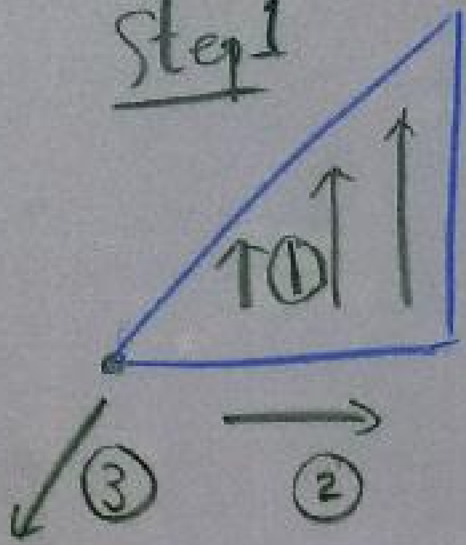
$z = G(x)$

Step 3 $\quad dx \longrightarrow \displaystyle\int_{0}^{1} dx$

$$\text{Ans} = \int_{x=0}^{1} \int_{z=0}^{1-x} \int_{y=x+z}^{1} 1\, dy\, dz\, dx$$

Case ② $dz\,dy\,dx$

<u>Step 1</u>



$$dz: \int_{(B)}^{(D)} dz = \int_{z=0}^{y-x} dz$$

$(B):\ z = F_1(x,y) = 0$

$(D):\ z = F_2(x,y) = y - x$

<u>Step 2</u> $$dy: \int_{(B,D)}^{(C)} \int_{0}^{y-x} dz = \int_{y=x}^{1} \int_{z=0}^{y-x} dz\, dy$$

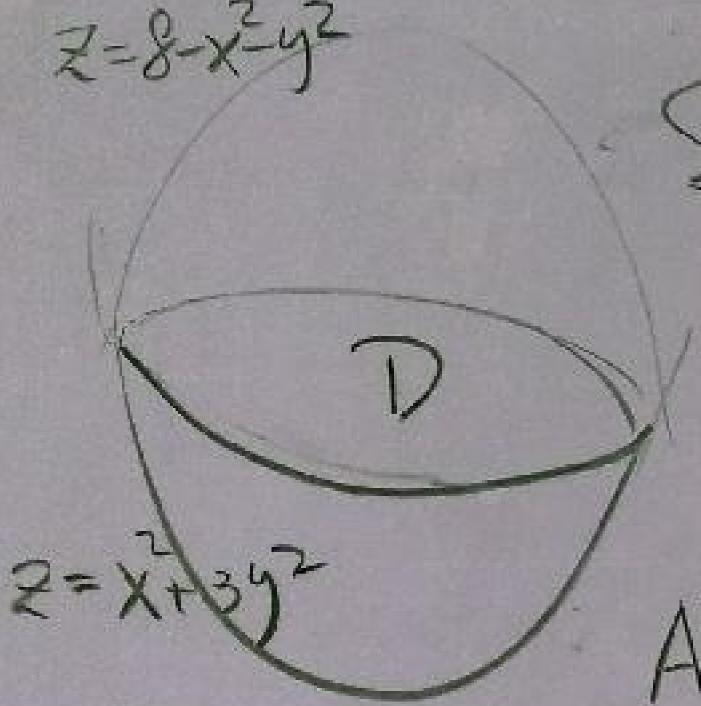
$(C):\ y = G_1(x) = 1$

$(B,D)\begin{cases} z = 0 \\ y = x + z \end{cases},\ y = G_2(x) = x$

<u>Step 3</u> $dx \rightarrow \int_0^1 dx$

<u>Ans:</u> $$\int_{x=0}^{1} \int_{y=x}^{1} \int_{z=0}^{y-x} dz\, dy\, dx$$

Example Find the volume of the domain $x^2+3y^2 \le z \le 8-x^2-y^2$



Sol: Obviously ① $= dz$

i.e. $dz\,dx\,dy$ or $dz\,dy\,dx$

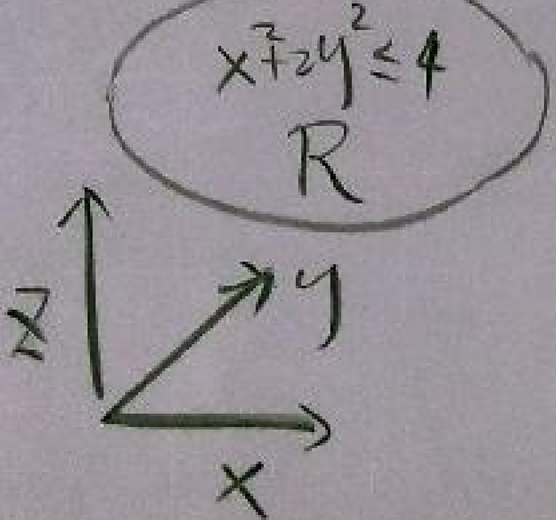
Example: take $dz\,dy\,dx$

After dz is done, we need to find region for $dy\,dx$

$$D = \{x^2+3y^2 \le z \le 8-x^2-y^2\}$$

$$\Rightarrow R = \{x^2+3y^2 \le 8-x^2-y^2\}$$

$$= \{x^2+2y^2 \le 4\}$$



$$V = \iiint_D 1 \, dV$$

$$= \iint_R \int_{x^2+3y^2}^{8-x^2-y^2} 1 \, dz \; dy\, dx.$$

dy: $$\int_{y=G_1(x)}^{G_2(x)} \left(\int_{x^2+3y^2}^{8-x^2-y^2} dz \right) dy$$

$y=G_1(x)$
$y=G_2(x)$ $\Rightarrow$ $x^2+2y^2=4$

$y = G_2(x)$

$y = G_1(x)$

$$G_{1,2}(x) = \pm\sqrt{\frac{4-x^2}{2}}$$

$$dz\,dy \Rightarrow \int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{z=x^2+3y^2}^{8-x^2-y^2} dz\,dy \quad (*)$$

$$\text{Ans} = \int_{x=-2}^{2} (*)\, dx$$