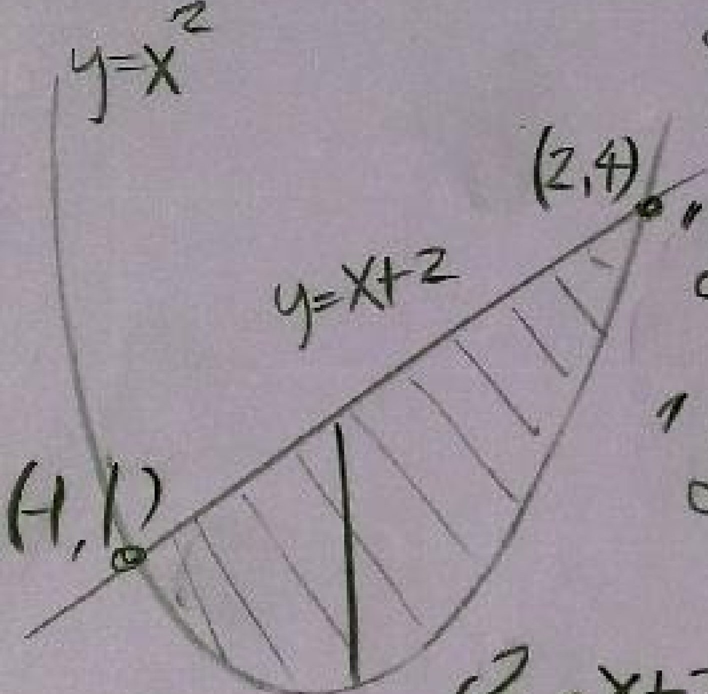
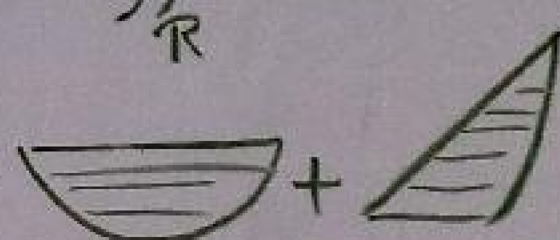
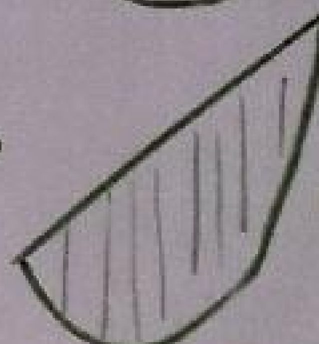
Area by double integral

Example: Find area enclosed by $y=x^2$ and $y=x+2$



Sol: $A=\iint_R 1\, dA$

"dxdy" $\rightarrow$ 

"dydx" $\rightarrow$  (simpler)

$$\text{Area}=\int_{x=-1}^{2}\int_{y=x^2}^{x+2} dy\, dx=\int_{-1}^{2} x+2-x^2\, dx$$

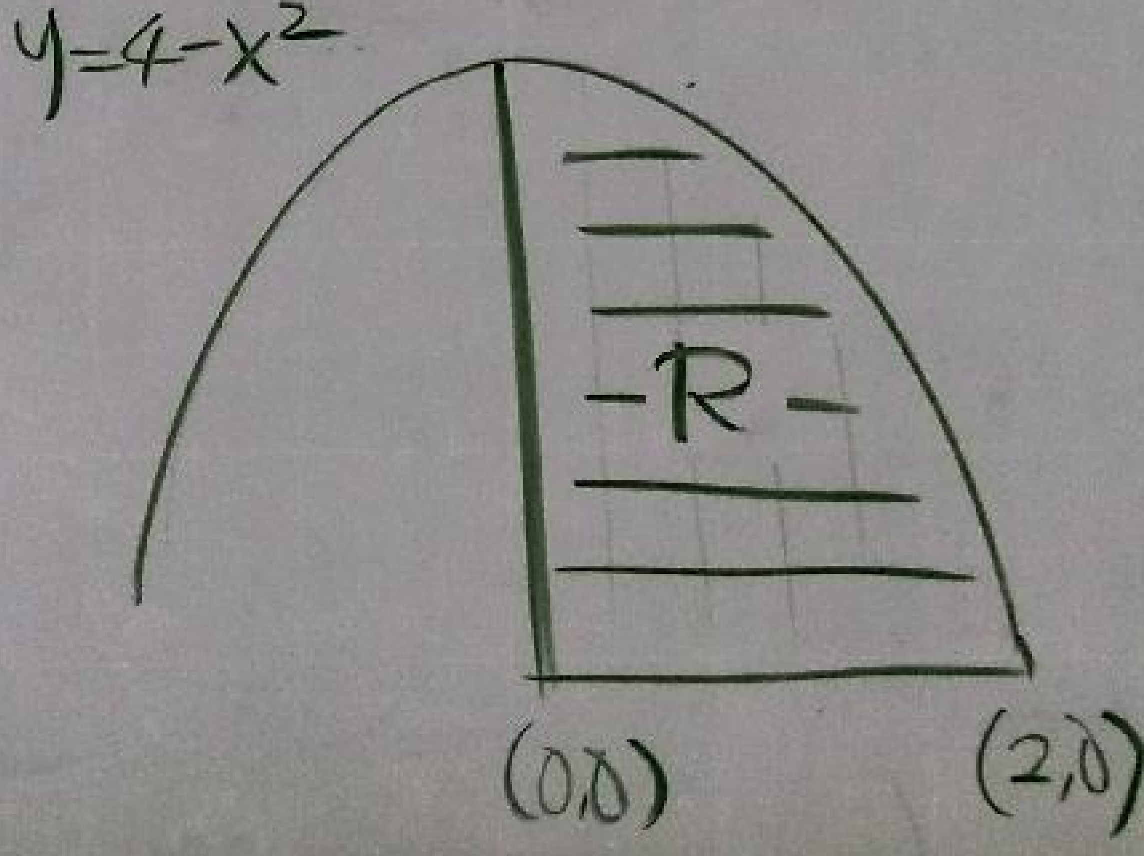
$$=\frac{x^2}{2}+2x-\frac{x^3}{3}\Big|_{-1}^{2}=\frac{9}{2}$$

Example $\displaystyle\int_0^2 \underbrace{\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy}_{\text{difficult}}\,dx$

Sol Try $dx\,dy$ instead

Note $\neq \displaystyle\int_0^{4-x^2}\int_0^2 \cdots\; dx\,dy$

Step 1 plot R



Step 3 Find limits of integration for dx

left : $x=0$

Right: $y=4-x^2 \Rightarrow x=\sqrt{4-y}$

Step 4

$$A=\int_{y=0}^{4}\int_{0}^{\sqrt{4-y}}\frac{xe^{2y}}{4-y}\,dx\,dy$$

$$=\int_{0}^{4}\frac{e^{2y}}{4-y}\left(\int_{0}^{\sqrt{4-y}}x\,dx\right)dy$$

$$=\int_{0}^{4}\frac{e^{2y}}{4-y}\left(\frac{x^2}{2}\right)\Bigg|_{x=0}^{\sqrt{4-y}}dy$$

$$=\int_{0}^{4}\frac{e^{2y}}{4-y}\cdot\frac{4-y}{2}\,dy=\frac{e^{2y}}{4}\Bigg|_{0}^{4}=\frac{e^{8}-1}{4}$$

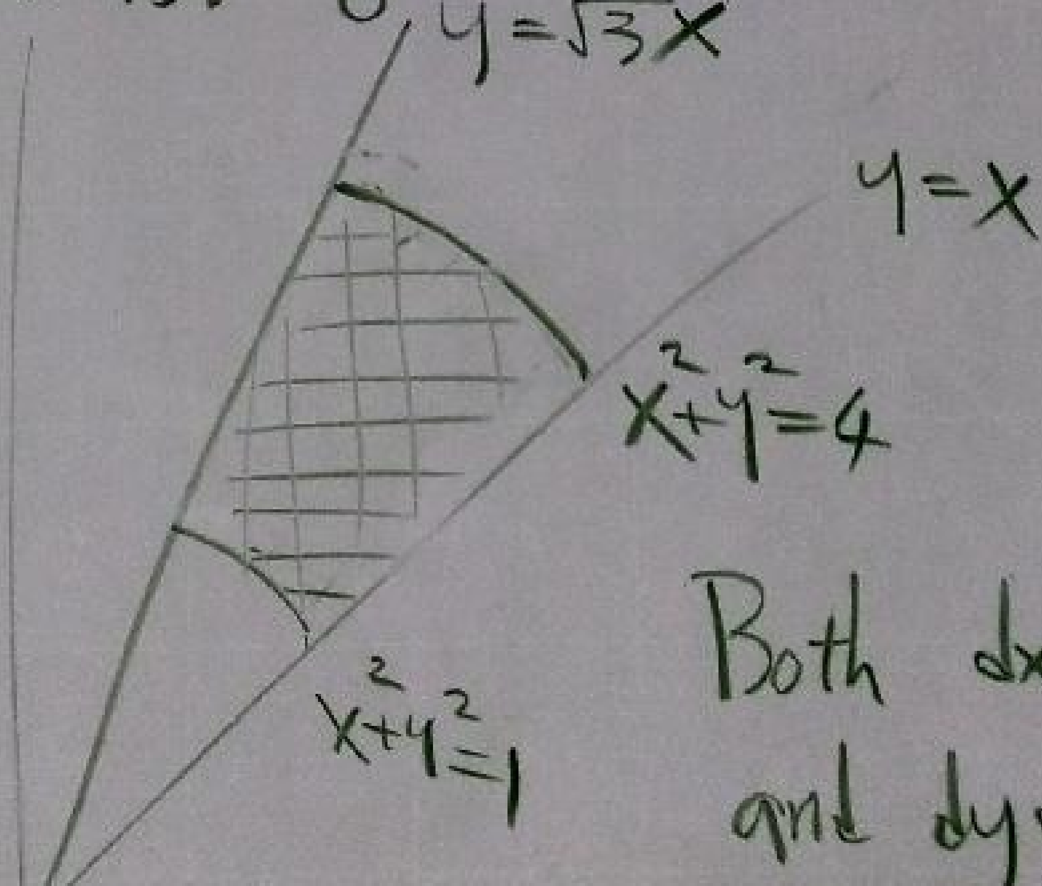
Integration in polar Coordinate

Example $\iint_R f(x,y)\,dA$

R = region bounded by

$$y = x \quad , \quad x^2+y^2=1$$
$$y = \sqrt{3}x \quad , \quad x^2+y^2=4$$

in 1st quadrant



Both dx dy and dy dx are complicated

In this example

$$R = \left\{ 1 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3} \right\}$$

So $dr\,d\theta$ or $d\theta\,dr$

are more convenient.

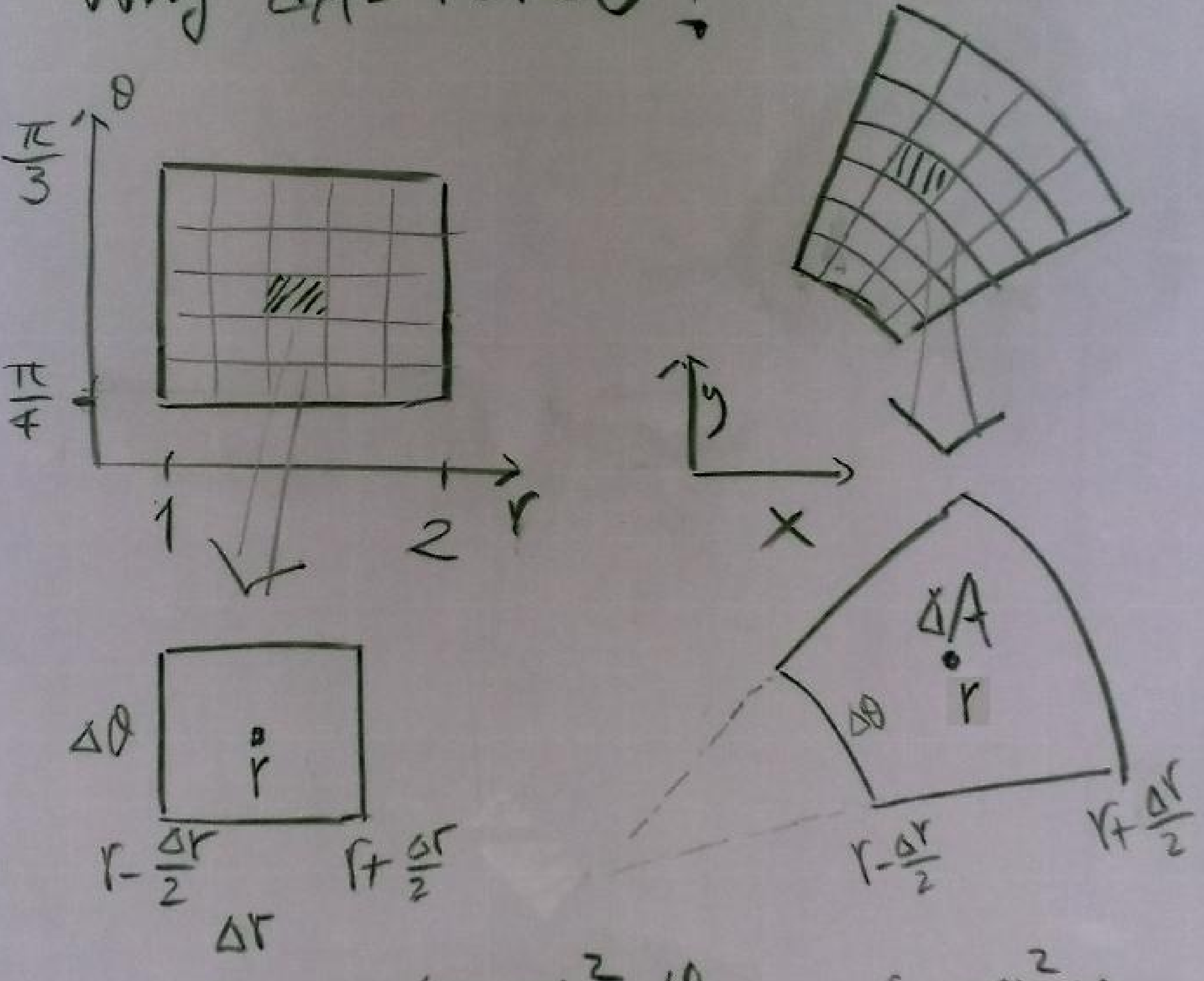
$$I = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{3}} \int_{r=1}^{2} f(r\cos\theta, r\sin\theta)\, dA$$

$$dA \overset{?}{=} (dr, d\theta)$$

<u>Ans</u>: $dA = r\,dr\,d\theta$

$\neq dr\,d\theta$

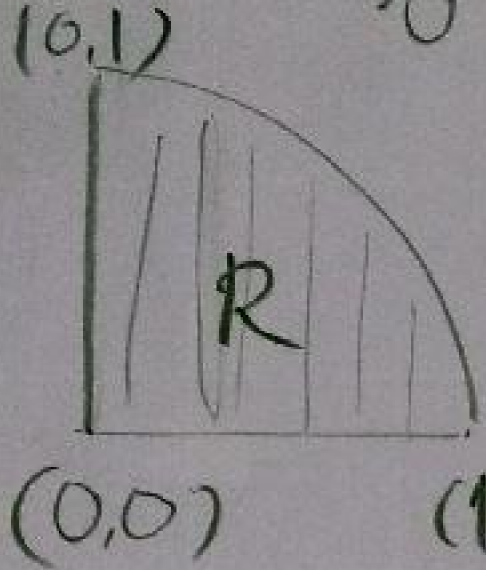
Why $dA = r\,dr\,d\theta$?



$$\Delta A = \pi\left(r+\frac{\Delta r}{2}\right)^2 \cdot \frac{\Delta\theta}{2\pi} - \pi\left(r-\frac{\Delta r}{2}\right)^2 \frac{\Delta\theta}{2\pi}$$

$$= \frac{\Delta\theta}{2}\left(\left(r+\frac{\Delta r}{2}\right)^2 - \left(r-\frac{\Delta r}{2}\right)^2\right) = r\,\Delta r\,\Delta\theta$$

Example $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$



$= \int_0^1 \left(x^2\sqrt{1-x^2} + \frac{\sqrt{1-x^2}^{\,3}}{3} \right) dx$

$=$ difficult

$\iint dx\,dy$ is similarly difficult

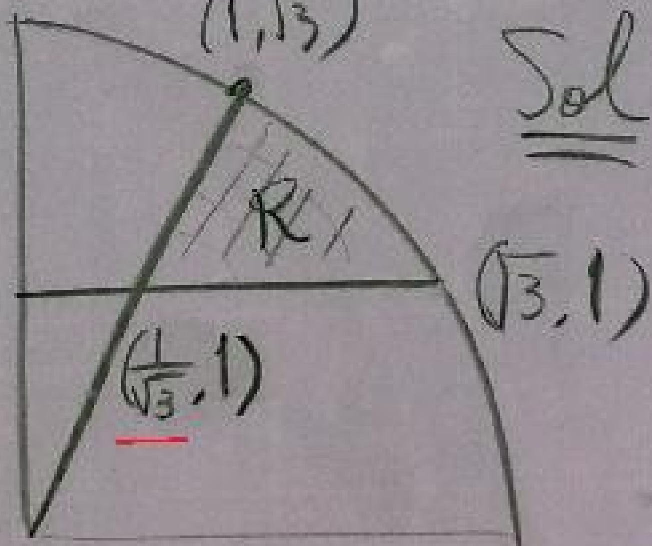
Polar coordinate: $R = \{ 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2} \}$

$$\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} (r^2)\, r\,dr\,d\theta$$

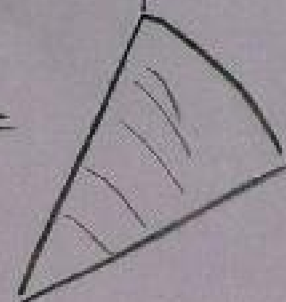
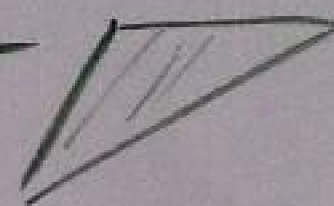
$$= \int_0^{\frac{\pi}{2}} \left. \frac{r^4}{4} \right|_{r=0}^{1} d\theta = \frac{\pi}{8}$$

Example Find area bounded by

$y=\sqrt{3}x$, $y=1$, $x^2+y^2=4$ in 1st quadrant



Sol Simplest method

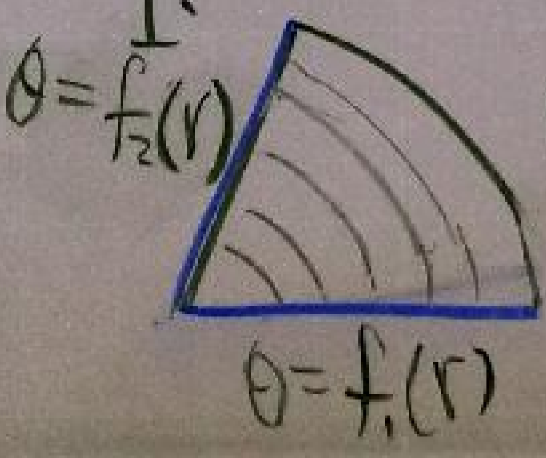
=  − 

$$= \frac{2^2}{2}\cdot\frac{\pi}{6} - \frac{1}{2}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)\cdot 1$$

We will show how to do this in polar coordinate

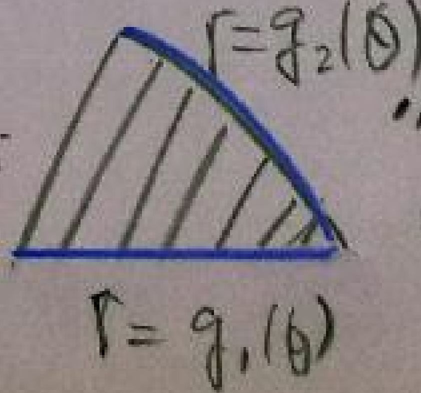
$$A=\iint_R 1\, r\,dr\,d\theta$$

I:



"$d\theta\, dr$"

II:



"$dr\, d\theta$"

$\mathrm{I}: R = \left\{ \frac{2}{\sqrt{3}} \le r \le 2,\ f_1(r) \le \theta \le f_2(r) \right\}$

$\theta = f_1(r) \iff y = 1$, $\theta = f_2(r) \iff y = \sqrt{3}x$

$r\sin\theta = 1$, $\underline{\theta = \sin^{-1}\left(\frac{1}{r}\right)}$, $r\sin\theta = \sqrt{3}\, r\cos\theta$, $\underline{\theta = \frac{\pi}{6}}$

$$\mathrm{I} = \int_{\frac{2}{\sqrt{3}}}^{2} \left(\int_{\theta = \sin^{-1}\left(\frac{1}{r}\right)}^{\frac{\pi}{6}} d\theta \right) r\,dr \to \text{difficult}$$

$\mathrm{II}: R = \left\{ \frac{\pi}{6} \le \theta \le \frac{\pi}{3},\ g_1(\theta) \le r \le g_2(\theta) \right\}$

$r = g_1(\theta) \iff y = 1$, $r = g_2(\theta) \iff x^2 + y^2 = 4$

$r\sin\theta = 1$, $\underline{r = \csc\theta}$ $\underline{r = 2}$

$$\mathrm{II} = \int_{\theta = \frac{\pi}{6}}^{\frac{\pi}{3}} \left(\int_{r = \csc\theta}^{2} r\,dr \right) d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{4 - \csc^2\theta}{2} \right) d\theta$$