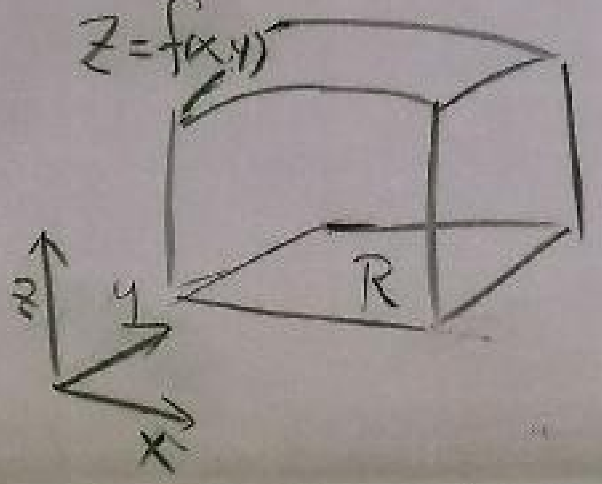
Double integral

Def: Let $R = [a,b] \times [c,d]$
$$= \{ a \le x \le b,\ c \le y \le d \}$$

$$\iint_R f(x,y)\, dA$$
(or $dx\,dy$)

= Signed volume between $z = f(x,y)$ and x-y plane over the region R



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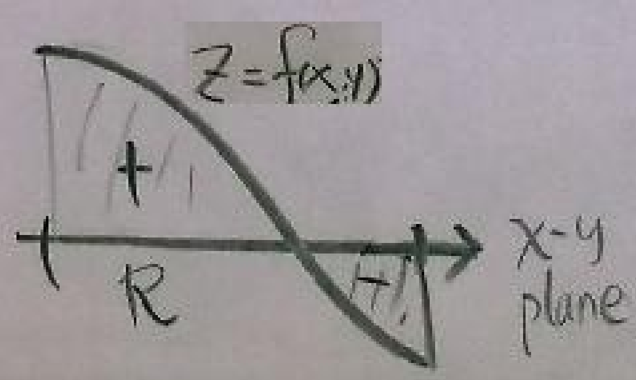
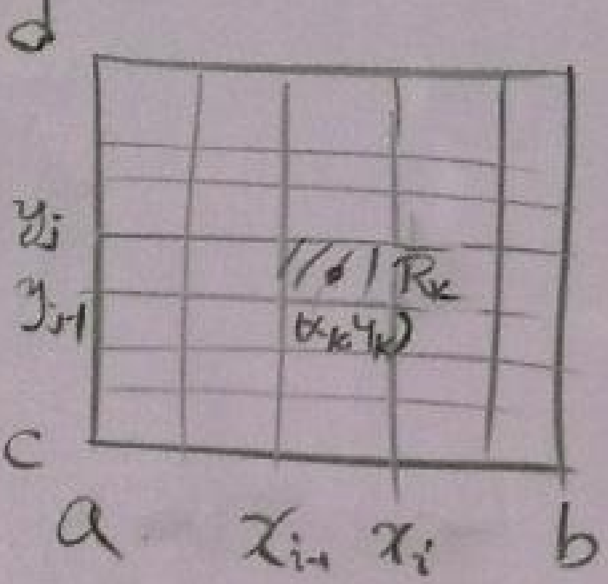
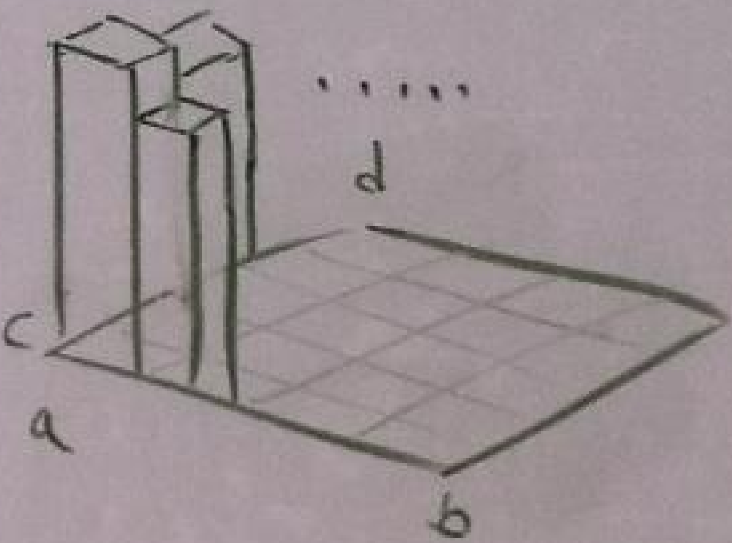


In other words

$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$





$$P = \left\{ \begin{matrix} a = x_0 < x_1 \cdots < x_M = b \\ c = y_0 < y_1 \cdots < y_N = d \end{matrix} \right\}$$

$$\|P\| = \max\{\Delta x_i, \Delta y_j\}$$

$(x_k, y_k) \in R_k$, $\Delta A_k =$ Area of R_k

Fubini's Theorem (1st form)

Let $R=[a,b]\times[c,d]$ and $f(x,y)$ is cont. on R

Then $\iint_R f(x,y)\,dA$

$$=\int_c^d\int_a^b f(x,y)\,dx\,dy=\int_a^b\int_c^d f(x,y)\,dy\,dx$$

Example: $R=[0,2]\times[0,1]$

$f(x,y)=4-x-y$, $\iint_R f(x,y)\,dA=?$

Ans. $\int_0^1\left(\int_0^2(4-x-y)\,dx\right)dy=\int_0^1\left(4x-\frac{x^2}{2}-xy\right)\Big|_{x=0}^{2}dy$

$$=\int_0^1(6-2y)\,dy=\left(6y-y^2\right)\Big|_{y=0}^{1}=5$$

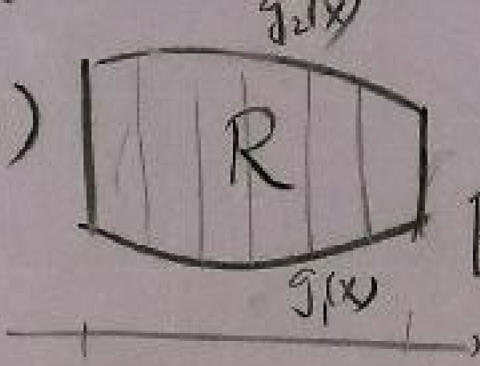
General region R

$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \underbrace{\sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k}$$

(only sum over those $R_k \subset R$)

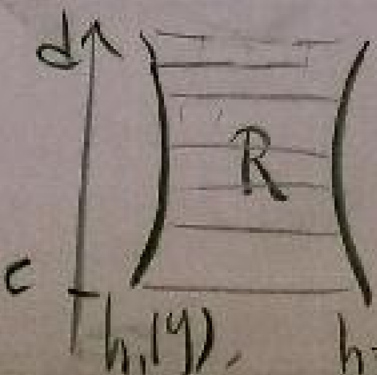
Special cases

(i)



$$R = \left\{ \begin{matrix} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{matrix} \right\}$$

(ii)



$$R = \left\{ \begin{matrix} c \le x \le d \\ h_1(y) \le x \le h_2(y) \end{matrix} \right\}$$

Fubini's Thm: If f is cont on R

(i) If $R = \{a \le x \le b,\ g_1(x) \le y \le g_2(y)\}$

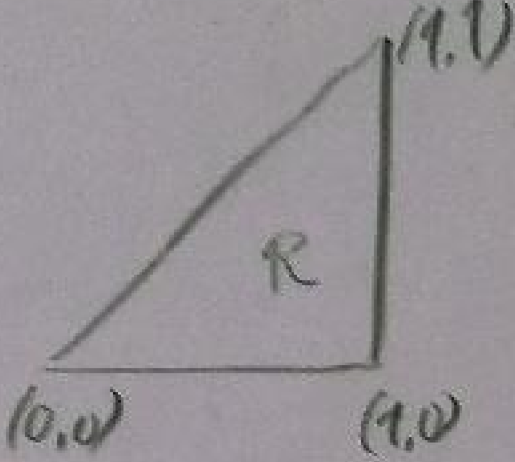
Then $$\iint_R f(x,y)\,dA = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(ii) If $R = \{c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

Then $$\iint_R f(x,y)\,dA = \int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Example R = region bounded by

$x=y$, $x=1$, $y=0$



$$\iint_R \frac{\sin x}{x}\, dA = ?$$

Sol $R = \{0 \le x \le 1,\ 0 \le y \le x\}$ (i)

$= \{0 \le y \le 1,\ y \le x \le 1\}$ (ii)

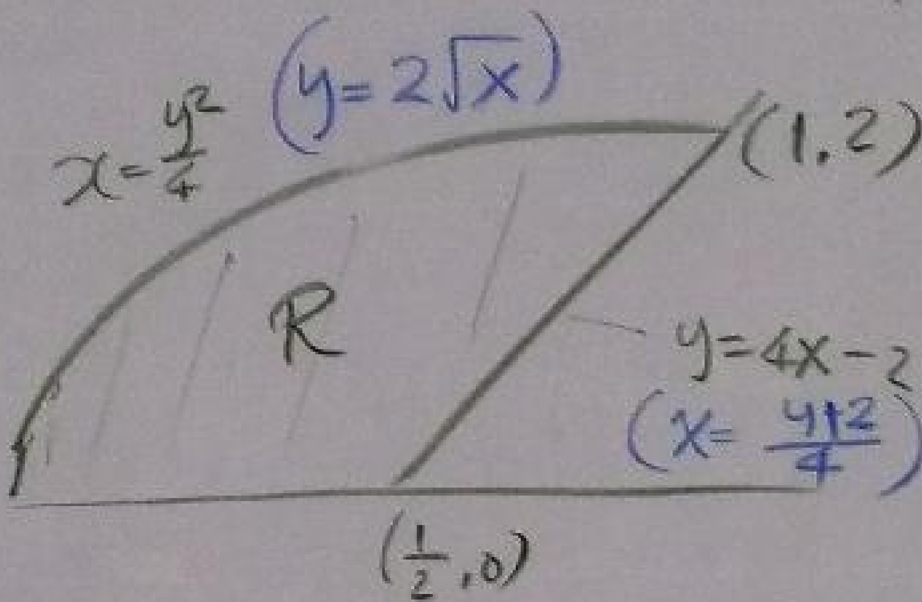
$$\text{(ii)} = \int_{y=0}^{1}\int_{x=y}^{1} \frac{\sin x}{x}\, dx\, dy = ?$$

$$\text{(i)} = \int_{x=0}^{1}\int_{y=0}^{x} \frac{\sin x}{x}\, dy\, dx$$

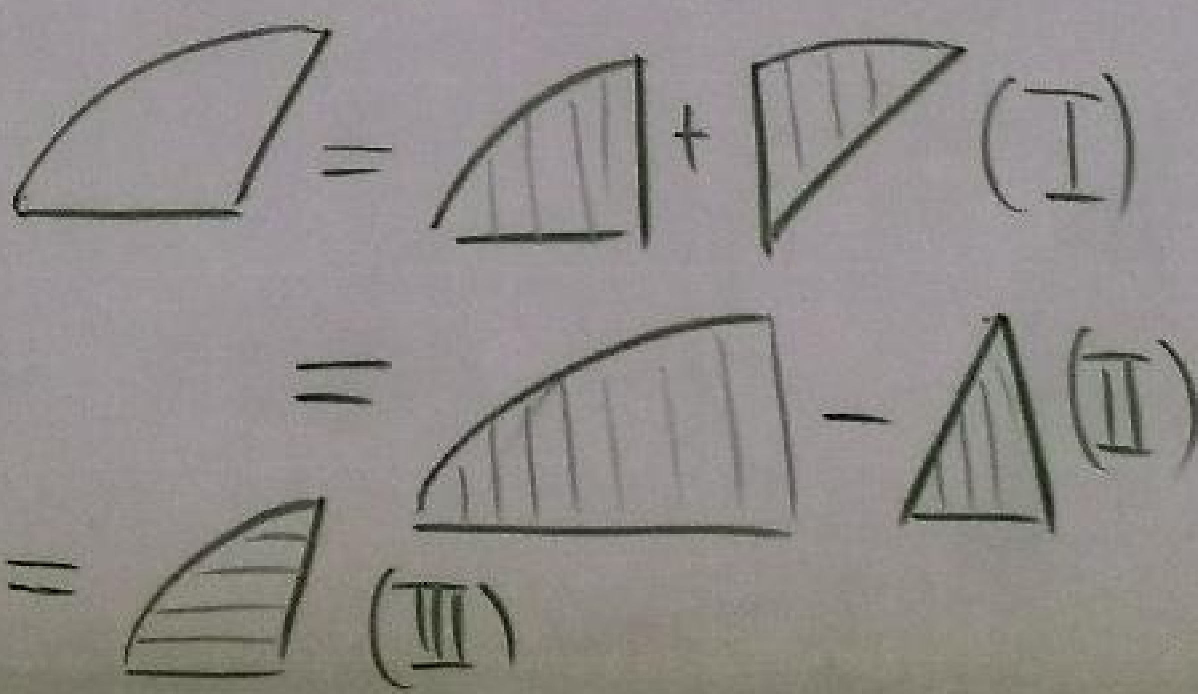
$$= \int_{x=0}^{1} \frac{\sin x}{x}\cdot(x-0)\, dx = \int_0^1 \sin x\, dx = 1-\cos 1$$

Example Area bounded by

$y=0, \quad y=4x-2, \quad x=\frac{y^2}{4}$



Sol $A=\iint_R 1\,dA$



(I) : exercise

$$(\text{II}) = \int_{x=0}^{1} 2\sqrt{x}\,dx - \frac{1}{2}\cdot\frac{1}{2}\cdot 2$$

$$= \frac{4}{3}x^{\frac{3}{2}}\Big|_0^1 - \frac{1}{2} = \frac{5}{6}$$

(III)

$$= \int_{y=0}^{2} \left(\frac{y+2}{4} - \frac{y^2}{4}\right) dy$$

$$= \int_0^2 \left(-\frac{y^2}{4} + \frac{y}{4} + \frac{1}{2}\right) dy$$

$$= -\frac{y^3}{12} + \frac{y^2}{8} + \frac{y}{2}\Big|_{y=0}^{2}$$

$$= -\frac{8}{12} + \frac{1}{2} + 1 = \frac{10}{12} = \frac{5}{6}$$