

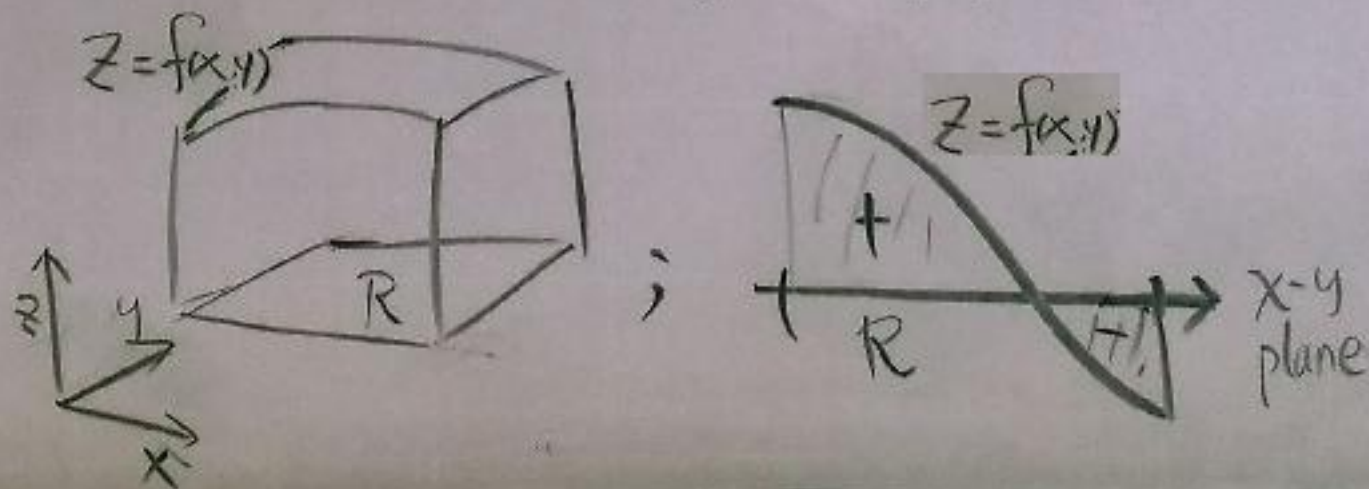
Double integral

Def: Let $R = [a, b] \times [c, d]$
 $= \{a \leq x \leq b, c \leq y \leq d\}$

$$\iint_R f(x, y) dA$$

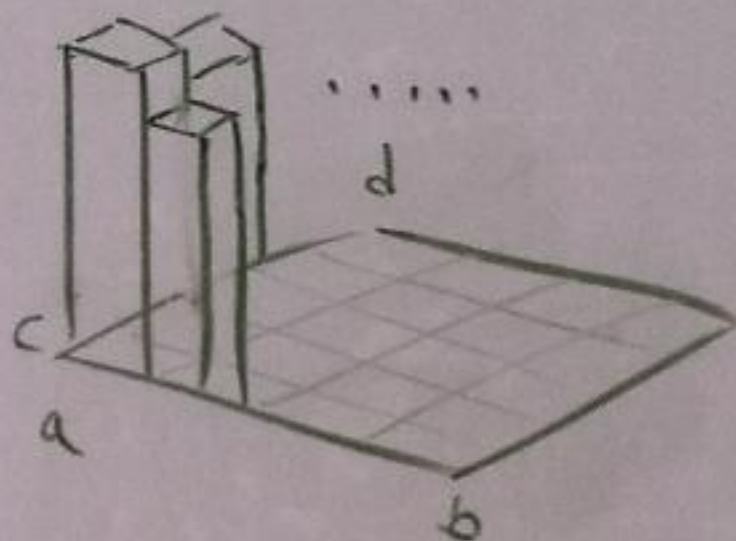
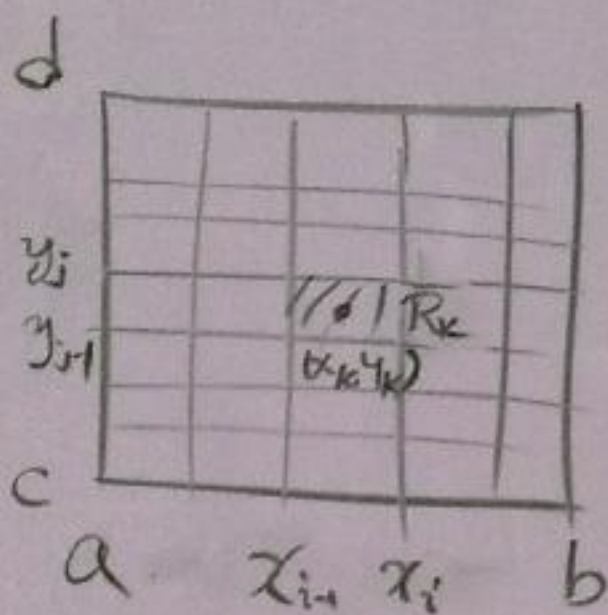
(or $dx dy$)

= Signed volume between
 $z = f(x, y)$ and x - y plane
over the region R



In other words

$$\iint_R f(x, y) dA = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$



$$P = \left\{ \begin{array}{l} a = x_0 < x_1 < \dots < x_N = b \\ c = y_0 < y_1 < \dots < y_N = d \end{array} \right\}$$

$$\|P\| = \max \{ \Delta x_i, \Delta y_j \}$$

$$(x_k, y_k) \in R_k, \Delta A_k = \text{Area of } R_k$$

Fubini's Theorem (1st form)

Let $R = [a, b] \times [c, d]$ and $f(x, y)$ is cont. on R

$$\begin{aligned} \text{Then } \iint_R f(x, y) dA \\ = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx \end{aligned}$$

Example: $R = [0, 2] \times [0, 1]$

$$f(x, y) = 4 - x - y, \quad \iint_R f(x, y) dA = ?$$

$$\begin{aligned} \text{Ans. } \int_0^1 \left(\int_0^2 (4 - x - y) dx \right) dy &= \int_0^1 \left(4x - \frac{x^2}{2} - xy \right) \Big|_{x=0}^2 dy \\ &= \int_0^1 (6 - 2y) dy = (6y - y^2) \Big|_0^1 = 5 \end{aligned}$$

General region R

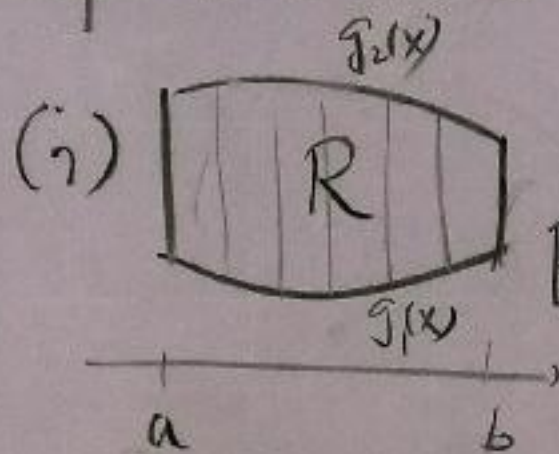


$$\iint_R f(x,y) dA$$

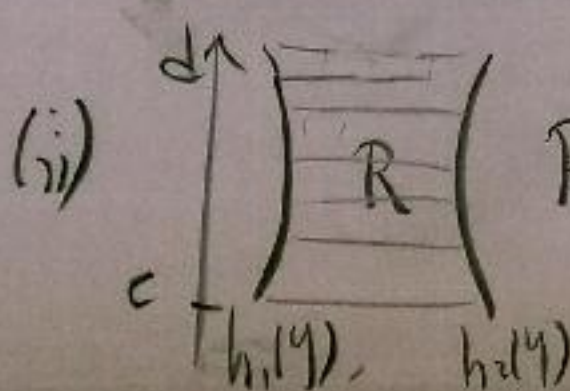
$$= \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \underbrace{f(x_k, y_k) \Delta A_k}_{R_k \subset R}$$

(only sum over those $R_k \subset R$)

Special cases



$$R = \left\{ \begin{array}{l} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{array} \right\}$$



$$R = \left\{ \begin{array}{l} c \leq y \leq d \\ h_1(y) \leq x \leq h_2(y) \end{array} \right\}$$

Fubini's Thm: If f is cont on R

(i) If $R = \{a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$

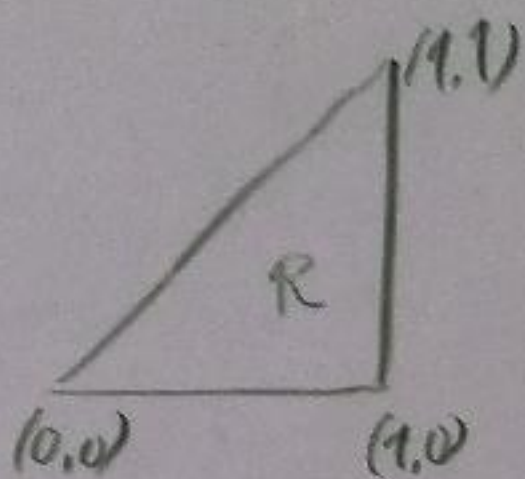
$$\text{Then } \iint_R f(x,y) dA = \int_{x=a}^b \int_{y=g_1(x)}^{g_2(x)} f(x,y) dy dx$$

(ii) If $R = \{c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$

$$\text{Then } \iint_R f(x,y) dA = \int_{y=c}^d \int_{x=h_1(y)}^{h_2(y)} f(x,y) dx dy$$

Example $R =$ region bounded by

$$x=y, x=1, y=0$$



$$\iint_R \frac{\sin x}{x} dA = ?$$

Sol $R = \{0 \leq x \leq 1, 0 \leq y \leq x\}$ (i)

$$= \{0 \leq y \leq 1, y \leq x \leq 1\}$$
 (ii)

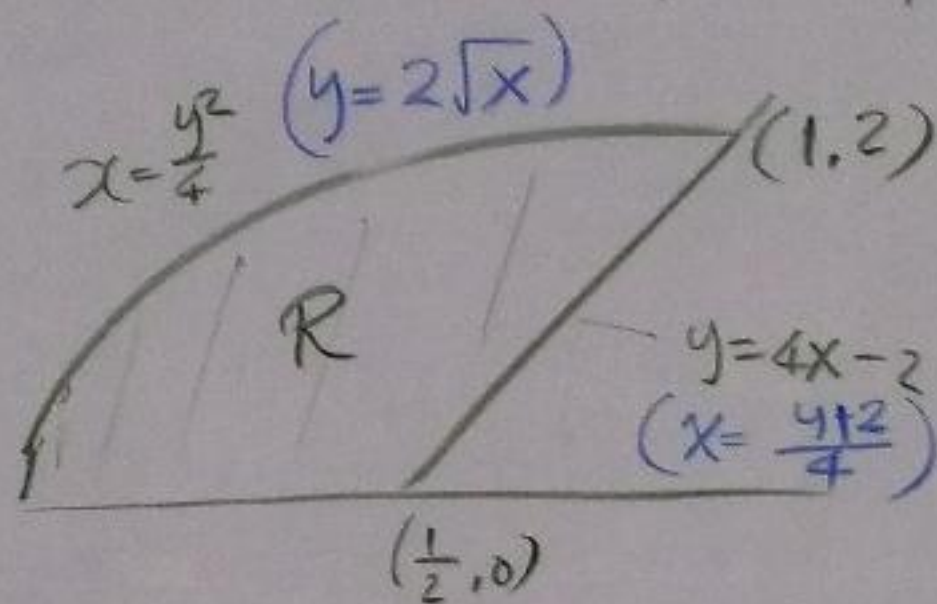
$$(ii) = \int_{y=0}^1 \int_{x=y}^1 \frac{\sin x}{x} dx dy = ?$$

$$(i) = \int_{x=0}^1 \int_{y=0}^x \frac{\sin x}{x} dy dx$$

$$= \int_{x=0}^1 \frac{\sin x}{x} \cdot (x-0) dx = \int_0^1 \sin x dx = 1 - \cos 1$$

Example Area bounded by

$$y=0, \quad y=4x-2, \quad x=\frac{y^2}{4}$$



$$\underline{\text{sol}} \quad A = \iint_R 1 \, dA$$

$$= \text{Diagram (I)} \quad \text{(I)}$$

Diagram (I) shows the region R decomposed into two parts: a region bounded by the x-axis, the parabola, and a vertical line at $x=1$, and a triangle bounded by the line $y=4x-2$, the vertical line $x=1$, and the x-axis.

$$= \text{Diagram (II)} \quad \text{(II)}$$

Diagram (II) shows the region R as the area under the parabola from $x=0$ to $x=1$, minus the area of the triangle bounded by the line $y=4x-2$, the x-axis, and the vertical line $x=1$.

$$= \text{Diagram (III)} \quad \text{(III)}$$

Diagram (III) shows the region R as a single shaded area bounded by the x-axis, the parabola, and the line $y=4x-2$.

(I): exercise

$$\begin{aligned} \text{(II)} &= \int_{x=0}^1 2\sqrt{x} dx - \frac{1}{2} \cdot \frac{1}{2} \cdot 2 \\ &= \frac{4}{3} x^{\frac{3}{2}} \Big|_0^1 - \frac{1}{2} = \frac{5}{6} \end{aligned}$$

(III)

$$\begin{aligned} &= \int_{y=0}^2 \left(\frac{y+2}{4} - \frac{y^2}{4} \right) dy \\ &= \int_0^2 \left(-\frac{y^2}{4} + \frac{y}{4} + \frac{1}{2} \right) dy \\ &= -\frac{y^3}{12} + \frac{y^2}{8} + \frac{y}{2} \Big|_{y=0}^2 \\ &= -\frac{8}{12} + \frac{1}{2} + 1 = \frac{10}{12} = \frac{5}{6} \end{aligned}$$