Lagrange multiplier

Case I

Find min/max of

$f(x,y)$ subject to

$g(x,y)=0$ 3 unknowns
3 equations

Case II

Find min/max of

$f(x,y,z)$ subject to

$g(x,y,z)=0$ 4 unknowns
4 equations

$$\begin{cases} g(x,y,z)=0 \\ \nabla f(x,y,z)=\lambda \nabla g(x,y,z) \end{cases}$$

Example:

Find closest point on $x^2-z^2-1=0$ to the origin $(0,0,0)$.

<u>Sol</u> $f(x,y,z)=x^2+y^2+z^2$

$g(x,y,z)=x^2-z^2-1=0$

$$\begin{cases} x^2-z^2-1=0 \\ 2x=\lambda 2x \\ 2y=0 \\ 2z=-\lambda 2z \end{cases}$$

$x(1-\lambda)=0,\ y=0,$
$z(1+\lambda)=0$

$1-\lambda=0,\ z=0$
or $1+\lambda=0,\ x=0$
or $x=0,\ z=0$ (No)

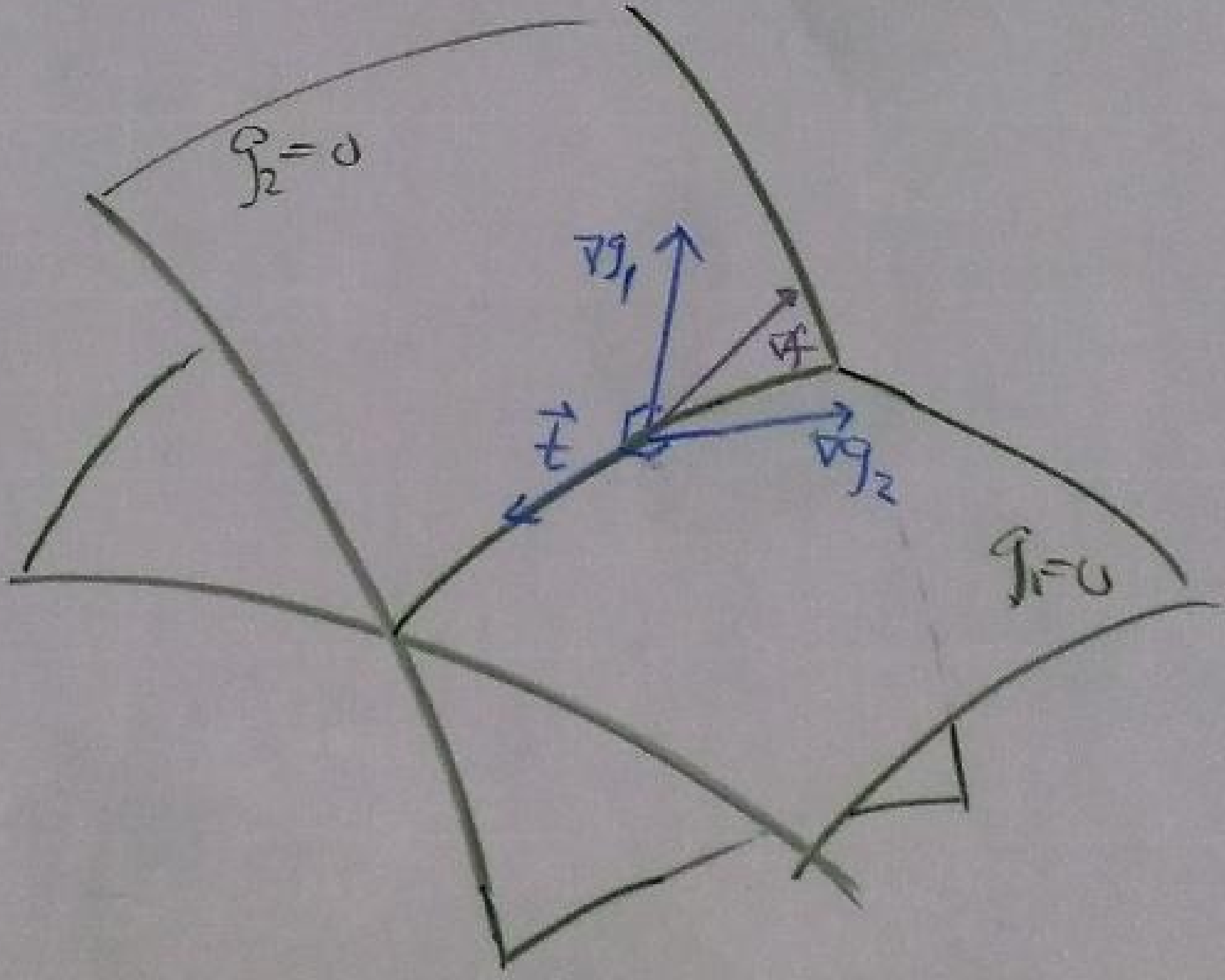
$\Rightarrow z=0,\ x=\pm 1$
or $x=0,\ z^2+1=0$ (No)

$\Rightarrow \underset{f=1}{\underline{(1,0,0)}},\ \underset{f=1}{\underline{(-1,0,0)}}$

Draw a figure
$\Rightarrow$ Both are absolute min

Case III Find min/max of $f(x,y,z)$ subject to $\Rightarrow$

$$\begin{cases} g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 \end{cases} \cdots (C)$$



Local extreme

$\Rightarrow \{f(x,y,z)=c_0\}$ is tangent to (C)

$\vec{t}$ = tangent vector of the curve (C)

$\vec{t} \perp \nabla g_1$

$\vec{t} \perp \nabla g_2$

$\vec{t} \perp \nabla f$

$\Rightarrow$ ∇f lies on the plane spanned by $\nabla g_1, \nabla g_2$

$$\Rightarrow \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

5 unknowns $x, y, z, \lambda_1, \lambda_2$

5 equations $\begin{cases} g_1 = 0 \\ g_2 = 0 \\ \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \end{cases}$

<u>Example</u> Find nearest point on $\begin{cases} x+y+z=1 \\ x^2+y^2=1 \end{cases}$ to the origin.

<u>Sol</u> $f(x,y,z) = x^2+y^2+z^2$

$$\begin{cases} x+y+z=1 & \text{— ①} \\ x^2+y^2=1 & \text{— ②} \\ 2x = \lambda_1 + \lambda_2 2x & \text{— ③} \\ 2y = \lambda_1 + \lambda_2 2y & \text{— ④} \\ 2z = \lambda_1 & \text{— ⑤} \end{cases}$$

From 5

3, 4 $\Rightarrow$ $x(1-\lambda_2) = z$

$y(1-\lambda_2) = z$

a $\lambda_2 = 1$, $z = 0$

$\Rightarrow$

or b $\lambda_2 \neq 1$, $x = y = \frac{z}{1-\lambda_2}$

Case a

1, 2 $\Rightarrow$ $x + y = 1$

$x^2 + y^2 = 1$

$\boxed{(x, y, z) = (1, 0, 0) \text{ or } (0, 1, 0)}$

Case b

2 $\Rightarrow$ $x = y = \pm\frac{1}{\sqrt{2}}$

1 $\Rightarrow$ $z = 1 \mp \sqrt{2}$

$(1,0,0), (0,1,0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1+\sqrt{2}\right)$

$f = \underbrace{1 \quad 1}_{\text{abs min}}, \quad 4-2\sqrt{2}, \quad 4+2\sqrt{2}$

abs min (local max) abs max

Taylor polynomial of $f(x,y)$

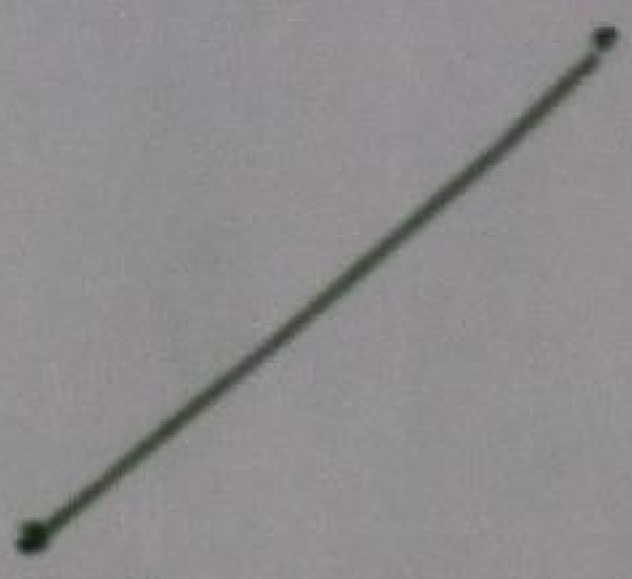
centered at (a,b):

$x = a+h, \quad y = b+k$

$f(x,y) = f(a+h, b+k)$

$= (\text{Taylor ?})$

$(x,y) = (a+h, b+k)$



(a,b)

$F(t) = f(a+th, b+tk)$

$t \in \mathbb{R}$

$$F(0) = f(a,b)$$
$$F(1) = f(a+h, b+k)$$

$$F(t) = F(0) + F'(0)t + \frac{F''(0)}{2}t^2$$
$$+ \cdots + \frac{F^{(n)}(0)}{n!}t^n + \frac{F^{(n+1)}(c)}{(n+1)!}t^{n+1}$$

Then let $t=1$

$n=1$, $F(1) = F(0) + F'(0) + \frac{F''(c)}{2}$

$$F'(t) = h\partial_x f + k\partial_y f$$
$$= (h\partial_x + k\partial_y) f(a+th, b+tk)$$
$$F''(t) = h^2\partial_x^2 f + 2hk\partial_x\partial_y f + k^2\partial_y^2 f$$
$$= (h\partial_x + k\partial_y)^2 f(a+th, b+tk)$$

$$F^{(n)}(t) = (h\partial_x + k\partial_y)^n f(a+th, b+tk)$$

$\underline{n=1}$

$$f(a+h, b+k) = f(a,b) + (h\partial_x + k\partial_y) f(a,b)$$

$$+ \frac{1}{2}(h\partial_x + k\partial_y)^2 f(a+ch, b+ck)$$

$0<c<1$

In general

$$f(a+h, b+k) = f(a,b) + \sum_{j=1}^{n} \frac{(h\partial_x + k\partial_y)^j}{j!} f(a,b)$$

$$+ R_n$$

$$R_n = \frac{(h\partial_x + k\partial_y)^{n+1}}{(n+1)!} f(a+ch, b+ck)$$

$0<c<1$