How to find interior local extrema of $f(x,y)$?

(Interior = NOT on boundary)

<u>Step 1</u>: Find all critical points (a,b), where $\nabla f(a,b) = \vec{0}$

<u>Step 2</u> $D = \left(f_{xy}^2 - f_{xx} f_{yy}\right)_{(a,b)}$

(i) If $f_{xx} > 0$, $D < 0$, local min

(ii) If $f_{xx} < 0$, $D < 0$, local max

(iii) If $D > 0$, saddle point

Example: find local extrema of x^2+y^2-4y+9

Ans: (method 1)

$f_x = 2x$, $f_y = 2(y-2)$

$\Rightarrow$ critical point $= (0,2)$

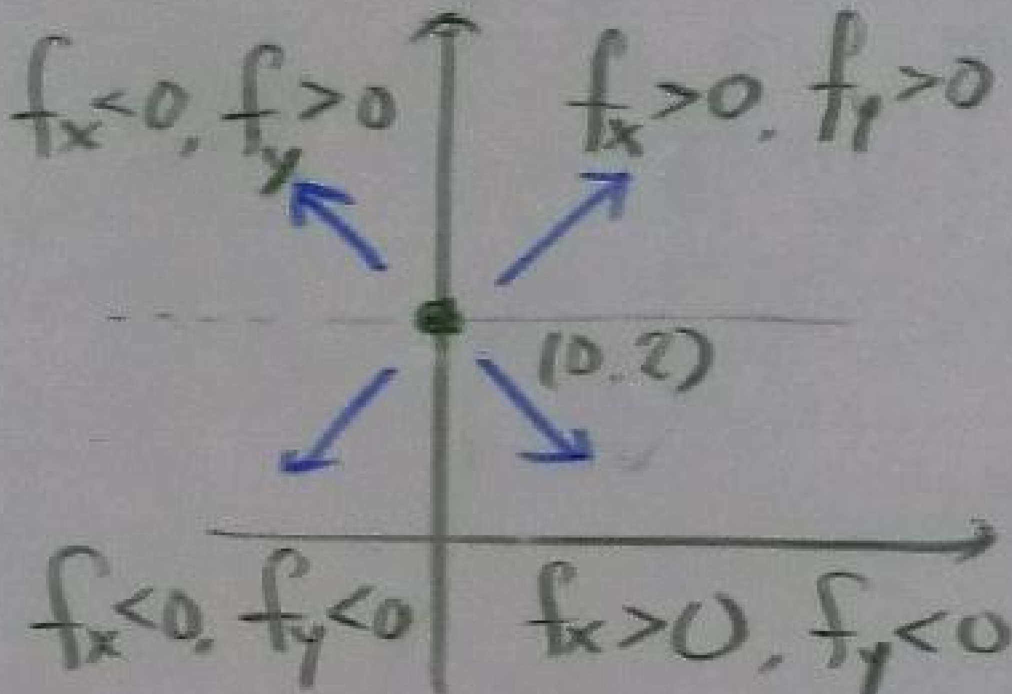
$f_{xx}(0,2) = 2$, $f_{xy}(0,2) = 0$

$f_{yy}(0,2) = 2$, $D = -4$

$\therefore$ local min

$(\because \Delta f \cong 2(\Delta x)^2 + 2(\Delta y)^2)$

Method 2 Gradient analysis near critical points



Rm

∇f points

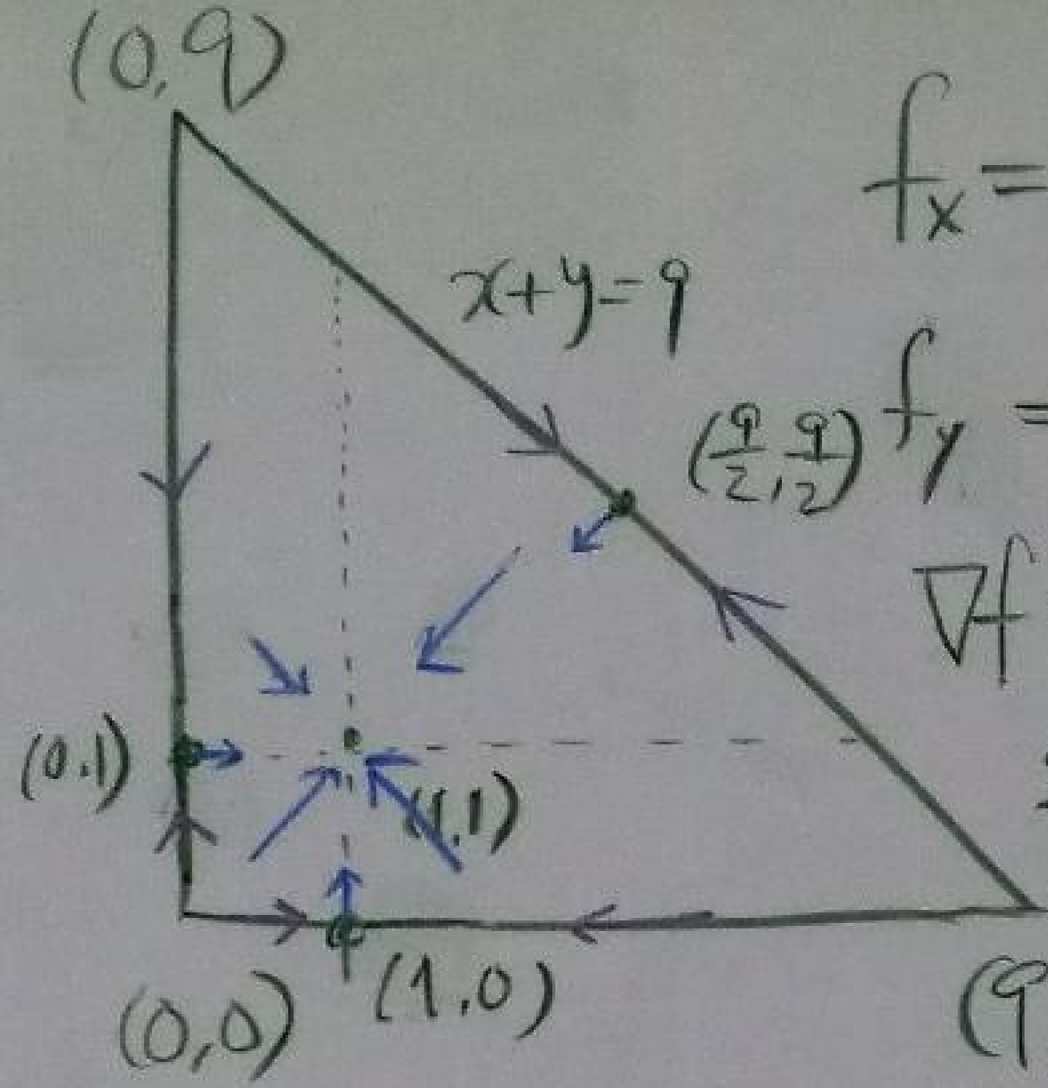
outward = local min

inward = local max

In this example, $(0,2)$ is also an absolute min sis

Saddle point

Example Find absolute min and absolute max of $2+2x+2y-x^2-y^2$ on the region bounded by $\begin{cases} x=0 \\ y=0 \\ y=9-x \end{cases}$



$f_x = -2(x-1)$

$f_y = -2(y-1)$

$\nabla f = \vec{0} \Rightarrow (1,1)$

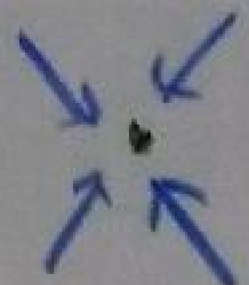
2nd derivative test

$(1,1)$ = local max

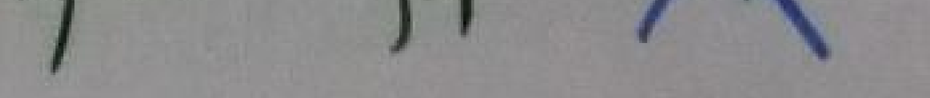
<u>Method 1</u>. Compare $(1,1)$ with all values on the boundary

<u>Method 2</u>

$f_x \begin{matrix} >0 \\ <0 \end{matrix}$ on $\begin{matrix} x<1 \\ x>1 \end{matrix}$, $f_y \begin{matrix} >0 \\ <0 \end{matrix}$ on $\begin{matrix} y<1 \\ y>1 \end{matrix}$:



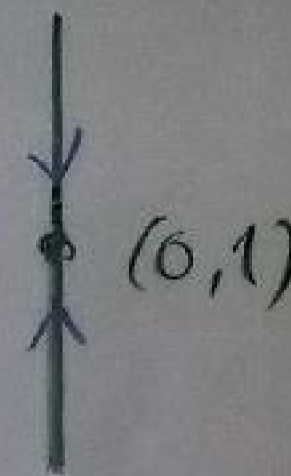
on $\{y=0\}$



$(1,0)$ = local max on $\{y=0\}$

Similarly on $\{x=0\}$

$(0,1)$ = local max on $\{x=0\}$



est

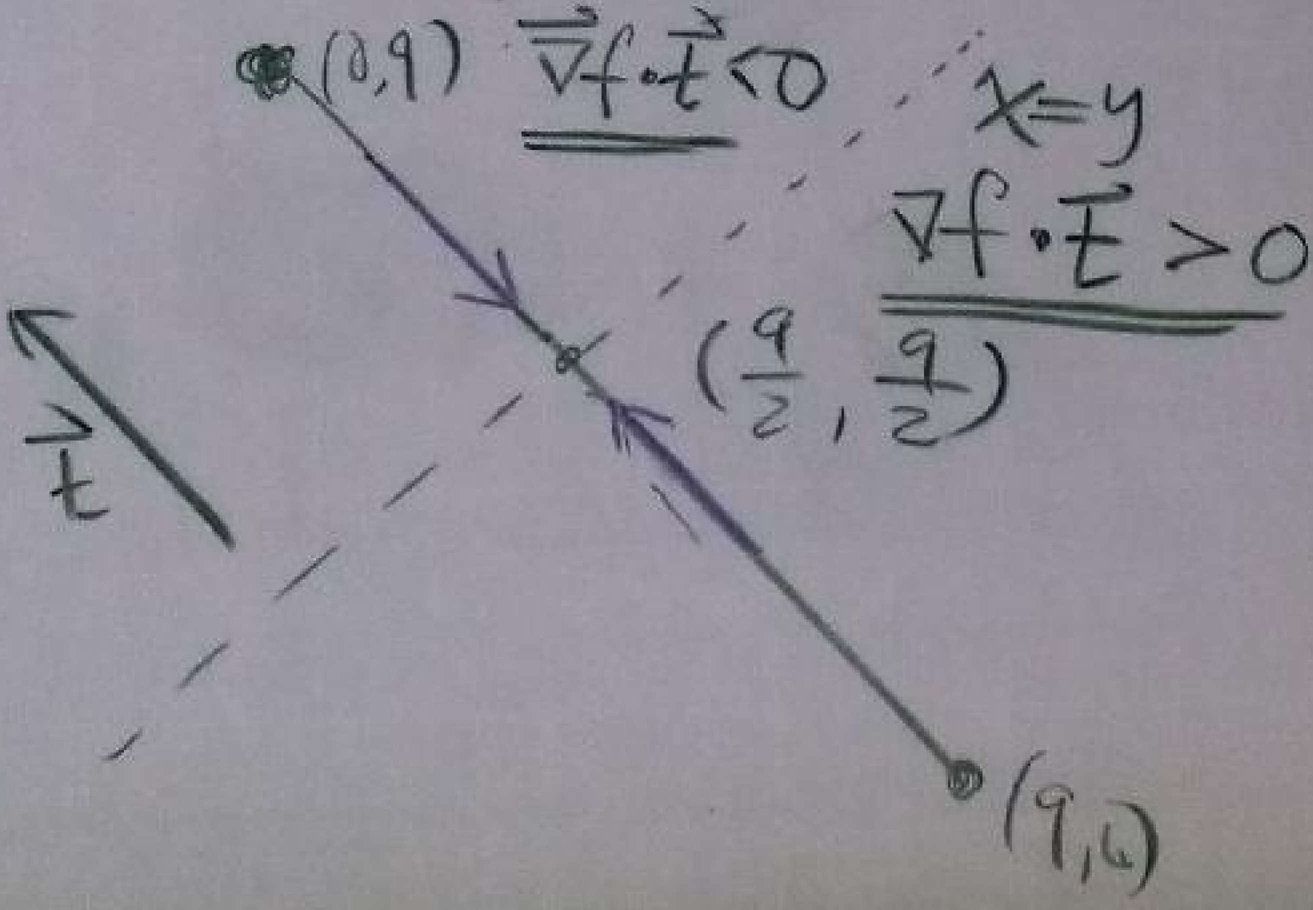
ax

on $x+y=9$

$\nabla f \cdot \vec{t} \quad \vec{t} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$= -2(x-1, y-1) \cdot \frac{(-1,1)}{\sqrt{2}}$

$= \sqrt{2}(x-y)$



,1) Absolute max

(1,1) (local max)

(1,0) (not local max)

Similarly for (0,1), $(\frac{9}{2},\frac{9}{2})$

$\Rightarrow$ (1,1) = <u>only</u> local max

= absolute max

<u>Absolute min</u>:

(0,0): , (0,9):

(9,0): , all 3 are

local min on bdry:

No interior local min

$\therefore$ Compare $f(0,0)$, $f(9,0)$, $f(0,9)$ to find absolute min.

Constrained optimization and Lagrangian multiplier

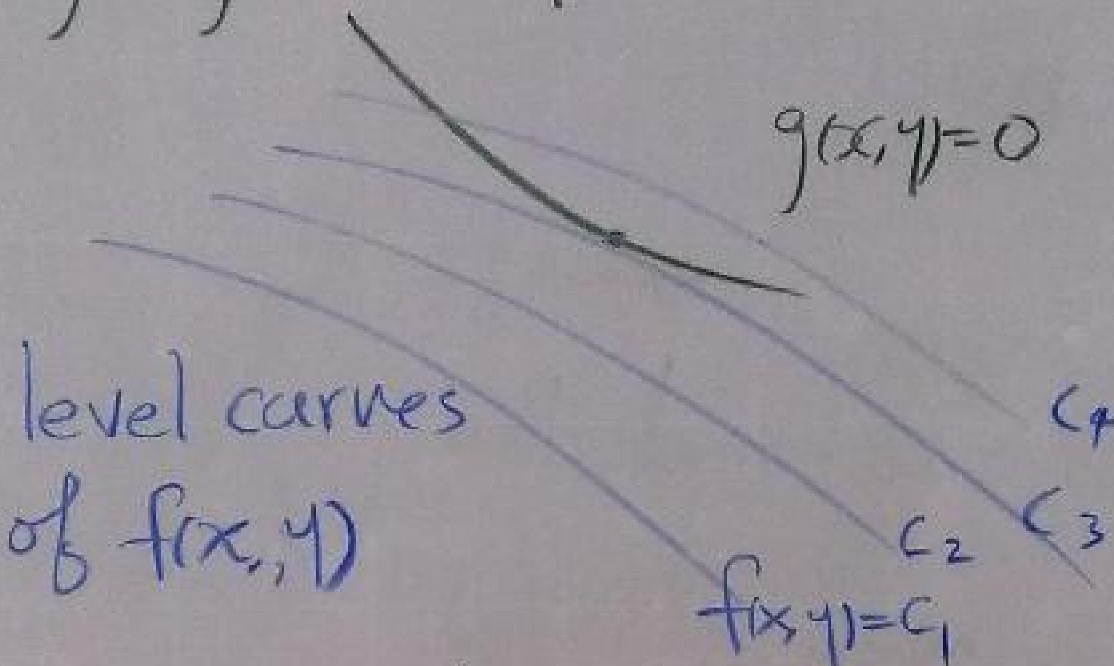
Goal: find local extreme points of $f(x,y)$ subject to the constraint $g(x,y)=0$

Example find the nearest point to the origin on $(x-\frac{1}{2})^2+\frac{y^2}{4}=1$

i.e. $f(x,y)=x^2+y^2$

$g(x,y)=(x-\frac{1}{2})^2+\frac{y^2}{4}-1$

Lagrange multiplier



Local extreme on $\{g(x,y)=0\}$

occurs when

$\{g(x,y)=0\}$ and $\{f(x,y)=c\}$

are tangent to each other.

i.e. when $\nabla f \parallel \nabla g$

or $\nabla f = \lambda \nabla g$ for some unknown λ

unknowns (x_0, y_0) (extrema point)

λ: Lagrangian multiplier

equations:

$$\begin{cases} g(x_0, y_0) = 0 \\ f_x(x_0, y_0) = \lambda g_x(x_0, y_0) \\ f_y(x_0, y_0) = \lambda g_y(x_0, y_0) \end{cases} \quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} \nabla f = \lambda \nabla g$$

In this example

$$\begin{cases} (x_0 - \frac{1}{2})^2 + \frac{y_0^2}{4} = 1 & \text{—} ① \\ 2x_0 = \lambda\, 2(x_0 - \frac{1}{2}) & \text{—} ② \\ 2y_0 = \lambda \frac{y_0}{2} & \text{—} ③ \end{cases}$$

From ③ $\Rightarrow$ $y_0 = 0$ or $\lambda = 4$

Case A: $y_0 = 0$

① $\Rightarrow$ $x_0 = \frac{3}{2}$ or $\frac{-1}{2}$

② $\Rightarrow$ $\lambda = 3$ or $\frac{1}{2}$

Case B $\lambda = 4$

$$② \Rightarrow x_0 = \frac{2}{3}$$

$$③ \Rightarrow y_0 = \frac{\pm\sqrt{35}}{3}$$

$$\left(\frac{3}{2}, 0\right), \underline{\left(\frac{-1}{2}, 0\right)} \underbrace{\left(\frac{2}{3}, \frac{\pm\sqrt{35}}{3}\right)}$$

local min   abs min   abs max
=nearst point

$$f = \frac{9}{4}, \frac{1}{4}, \frac{13}{3}, \frac{13}{3}$$