Example $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$

Find approximate value of $f(3.1, 1.9)$ and estimate the error of this appx.

(1) approximation near $(3,2)$

$L(x,y) = f(3,2) + f_x(3,2)\,\Delta x + f_y(3,2)\,\Delta y$

$f(3,2) = 8$, $f_x(3,2) = 4$, $f_y(3,2) = -1$

$\Delta x = 3.1 - 3 = 0.1$, $\Delta y = 1.9 - 2 = -0.1$

approximate value $= L(3.1, 1.9)$

Rm $\Delta f \cong f_x(3,2)\,\Delta x + f_y(3,2)\,\Delta y$

$f(3.1, 1.9) - 8 \cong 4 \cdot (0.1) + -1 \cdot (-0.1)$

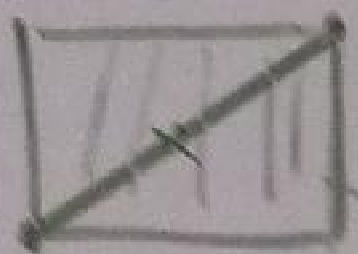
Ex

(N) Error estimate

$f_{xx} = 2,\ f_{xy} = -1,\ f_{yy} = 1$

$|f_{xx}|, |f_{xy}|, |f_{yy}| \le 2$

$(x_0+\Delta x,\ y_0+\Delta y)$



$\to |f_{xx}|, |f_{xy}|, |f_{yy}| \le 2$

$(x_0, y_0)$

$\therefore |\text{Error}| \le \frac{2}{2}\left(|\Delta x| + |\Delta y|\right)^2 = 0.04$

<u>Rm</u> $w = f(x, y, z)$   $* = x_0, y_0, z_0$

$L(x, y, z) = f(*) + f_x(*)\Delta x + f_y(*)\Delta y + f_z(*)\Delta z$

$|E| \le \frac{M}{2}\left(|\Delta x| + |\Delta y| + |\Delta z|\right)^2$, $|f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{yz}|, |f_{xz}| \le M$

Example: Volume of Cylinder

$V(r,h) = \pi r^2 h$

If $r_0 = 1\,m$, $h_0 = 5\,m$

$r_1 = r_0 + \Delta r$, $h_1 = h_0 + \Delta h$

$\Delta r = 0.1$ $\Delta h = -0.1$

$\Delta V = V(r_1, h_1) - V(r_0, h_0) \hat{=} ?$

$|E| \leq ?$

Ans: $V_r = 2\pi r h$, $V_h = \pi r^2$

$\Delta V \cong (2\pi r h_0)\Delta r + (\pi r_0^2)\Delta h$

$V_{rr} = 2\pi h$, $V_{rh} = 2\pi r$, $V_{hh} = 0$

$|V_{rr}| \leq 2\pi \max(h_0, h_1) = 2h_0 = 10\pi\, m$

$\leq M$ $|V_{rh}| \leq 2\pi r_1 = 2.2\pi\, m$ $M =$

$|E| \leq \frac{M}{2}(|\Delta r| + |\Delta h|)^2 = 0.2\pi\, m^3$

# Extreme values and Saddle points

How to find <u>local</u> min/max for $f(x,y)$ ? (assume all derivatives are cont.)

(i) At an local min/max $(x_0, y_0)$

$$\nabla f(x_0, y_0) = (0,0)$$

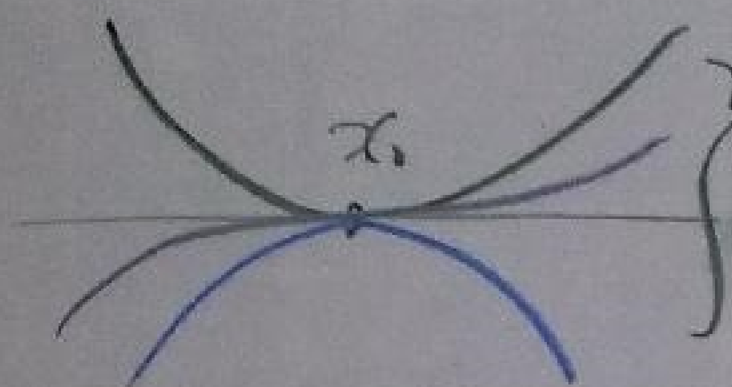
i.e. the tangent plane is horizontal (linear appx)

$$\because \Delta f = \nabla f(x_0, y_0) \cdot (\Delta x, \Delta y)$$

If $\nabla f(x_0, y_0) \neq (0,0)$, then

f increases along $(\Delta x, \Delta y) \parallel \nabla f(x_0, y_0)$
decreases $-\nabla f(x_0, y_0)$

In 1D case: $f'(x_0)=0$




It can be shown that

(See Equation (7), page 840)

$$\Delta f = \nabla f(x_0,y_0)\cdot(\Delta x,\Delta y) \quad \cdots\cdots \Delta_1$$

$$+\frac{1}{2}\left(f_{xx}(x_0,y_0)(\Delta x)^2 + 2f_{xy}(x_0,y_0)\,\Delta x\Delta y + f_{yy}(x_0,y_0)(\Delta y)^2\right) \quad \cdots\cdots \Delta_2$$

$$+ O\left((\Delta x)^3, (\Delta x)^2\Delta y, \Delta x(\Delta y)^2, (\Delta y)^3\right) \cdots \Delta_3$$

$\Delta_{1,2,3}$ = (linear, quadratic, cubic) in $(\Delta x, \Delta y, \Delta z)$

At a critical point $(\nabla f(x_0,y_0)=(0,0))$

$\Delta f = \Delta_1 + \Delta_2 + \Delta_3$

$\nabla f = 0 \Rightarrow \Delta_1 = 0$

leading order term $= \Delta_2$

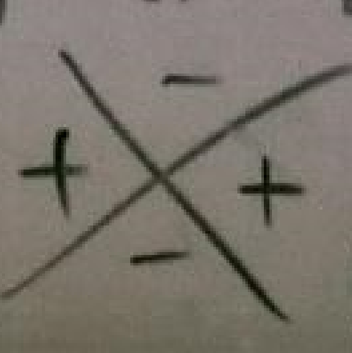
$$\Delta_2 = A(\Delta x)^2 + B\,\Delta x\,\Delta y + C(\Delta y)^2$$

$A = f_{xx}(x_0,y_0)$, $B = 2f_{xy}(x_0,y_0)$, $C = f_{yy}(x_0,y_0)$

$$\begin{cases} \Delta_2 > 0 \text{ for all } (\Delta x, \Delta y) \Leftrightarrow A > 0,\ B^2 - 4AC < 0 \\ \Delta_2 < 0 \text{ for all } (\Delta x, \Delta y) \Leftrightarrow A < 0,\ B^2 - 4AC < 0 \\ \Delta_2 \text{ can change sign } (+,-) \Leftrightarrow B^2 - 4AC > 0 \end{cases}$$

$\downarrow$

2nd derivative test




$(\Delta x, \Delta y)$ plane

$$\Delta_2 = 0 \iff A\left(\frac{\Delta x}{\Delta y}\right)^2 + B\frac{\Delta x}{\Delta y} + C = 0$$

$$\iff \frac{\Delta x}{\Delta y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

| | | $\frac{-B-\sqrt{\ }}{2A}$ | | $\frac{-B+\sqrt{\ }}{2A}$ | |
|---|---|---|---|---|---|
| $A>0$ | $+$ | | $-$ | | $+$ |
| $A<0$ | $-$ | | $+$ | | $-$ |