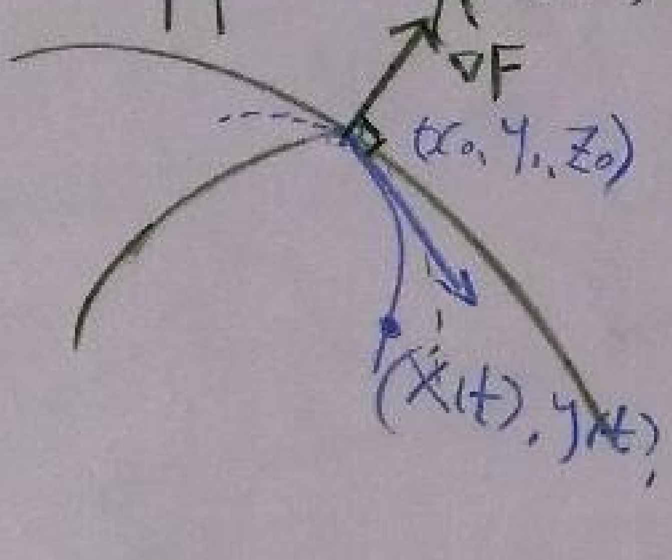
Tangent planes and Normal lines

$\{(x,y,z) \mid F(x,y,z)=c\}$

$=$ level surface of $F:\mathbb{R}^3 \longrightarrow \mathbb{R}$

Suppose $\{(x(t), y(t), z(t))\}$



$\subseteq$ level surface of F

and (x_0, y_0, z_0)

$=(x(t_0), y(t_0), z(t_0))$

$$F(x(t), y(t), z(t)) = c$$

$$\left.\frac{d}{dt}\right|_{t=t_0} \Rightarrow \nabla F(x(t_0), y(t_0), z(t_0)) \cdot (x'(t_0), y'(t_0), z'(t_0)) = 0$$

Since $(x'(t_0), y'(t_0), z'(t_0))$

is a tangent vector

of the surface at (x_0, y_0, z_0)

$$\Rightarrow \nabla F_{(x_0, y_0, z_0)} \perp \left(\begin{array}{c}\text{any tangent} \\ \text{vector at} \\ (x_0, y_0, z_0)\end{array}\right)$$

$$\Rightarrow \nabla F_{(x_0, y_0, z_0)} \perp \left(\begin{array}{c}\text{tangent} \\ \text{plane at} \\ (x_0, y_0, z_0)\end{array}\right)$$

$\Rightarrow$ Equation of normal line:

$$(x-x_0, y-y_0, z-z_0) \parallel \nabla F_{(x_0, y_0, z_0)}$$

Equation of tangent plane

$$(x-x_0, y-y_0, z-z_0) \cdot \nabla F(x_0,y_0,z_0) = 0$$

Example: Find the tangent plane and normal line of $x^2+\frac{y^2}{4}+\frac{z^2}{9}=3$ at $(1,2,3)$

$$F(x,y,z) = x^2+\frac{y^2}{4}+\frac{z^2}{9}-3$$

$$\nabla F_{(1,2,3)} = \left(2x, \frac{y}{2}, \frac{2z}{9}\right)_{(1,2,3)} = \left(2, 1, \frac{2}{3}\right)$$

tangent plane: $(x-1, y-2, z-3)\cdot\left(2,1,\frac{2}{3}\right)=0$

normal line: $x=1+2t,\ y=2+t,\ z=3+\frac{2}{3}t$

normal line

$(x-x_0, y-y_0, z-f(x_0,y_0))$

$\parallel (f_x(x_0,y_0), f_y(x_0,y_0), -1)$

$x(t) = x_0 + t f_x(x_0, y_0)$

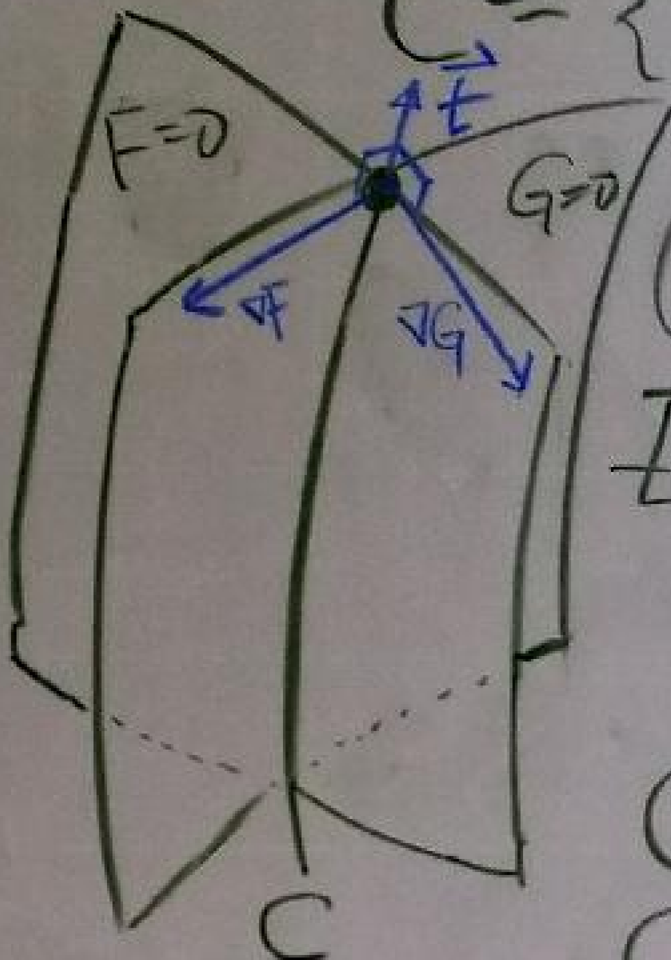
$y(t) = y_0 + t f_y(x_0, y_0)$

$z(t) = f(x_0, y_0) - t$

Example: Find tagent line

of $\begin{cases} x^2+y^2-2=0 \\ x+z-4=0 \end{cases}$

at $(1,1,3)$



$$C = \begin{cases} F(x,y,z) = 0 \\ G(x,y,z) = 0 \end{cases}$$

$(x_0, y_0, z_0) \in C$

$\vec{t}$ = tangent vector at (x_0, y_0, z_0)

$C \subseteq \{F(x,y,z)=0\}$

$C \subseteq \{G(x,y,z)=0\}$

$\therefore$ tangent vector of C is a tangent vector of $\{F=0\}$ also a tangent vector of $\{G=0\}$

$\vec{t} \perp \nabla F$, $\vec{t} \perp \nabla G$

$\Rightarrow \vec{t} \parallel \nabla F \times \nabla G$

In this example

$F = x^2 + y^2 - 2$, $G = x + z - 4$

$\nabla F_{(1,1,3)} = (2x, 2y, 0)_{(1,1,3)} = (2,2,0)$

$\nabla G_{(1,1,3)} = (1,0,1)$

tangent line

$(X-1, Y-1, Z-3)$

$/\!/ (2,2,0) \times (1,0,1) = \begin{pmatrix} i & j & k \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

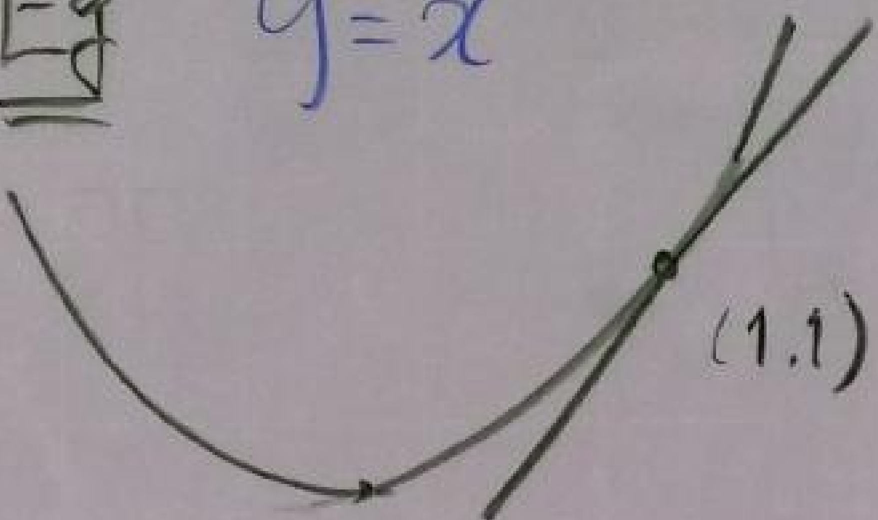
$= (2, -2, -2)$

$x = 1 + 2t$
$y = 1 - 2t$
$z = 3 - 2t$

Linearization
= approximate by a linear function

<u>Eg</u> $y = x^2$

$\frac{y-1}{x-1} = (x^2)'_{x=1} = 2$



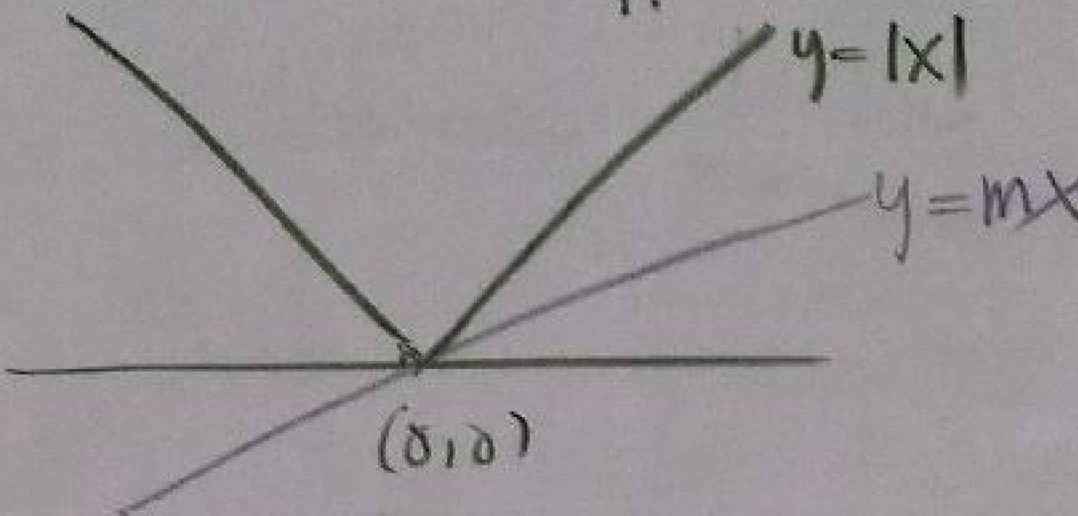
linear approximation

$y = 1 + 2(x-1)$ is the linearization of $y = x^2$ near $(1,1)$

$$\lim_{x \to 1} \frac{x^2 - (1+2(x-1))}{x-1} = 0 \quad (f'(1) \text{ exists})$$

Example $y=|x|$

has no linear app near $(0,0)$



$$\lim_{x\to 0}\frac{|x|-mx}{x-0}\neq 0$$

for any $m\in\mathbb{R}$

($f'(0)$ does not exist)

For $z = f(x, y)$

If f is diff. at (x_0, y_0)

$$f(x,y) = \underline{f(x_0, y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$
$$+ \varepsilon_1 (x-x_0) + \varepsilon_2 (y-y_0)$$

where $\lim\limits_{(x,y)\to(x_0,y_0)} \varepsilon_1, \varepsilon_2 = 0$

Rm: $\varepsilon_1(x-x_0) + \varepsilon_2(y-y_0)$

$$= \Big(\underbrace{\varepsilon_1 \frac{(x-x_0)}{\sqrt{}} + \varepsilon_2 \frac{(y-y_0)}{\sqrt{}}}_{=\varepsilon} \Big) \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$\lim\limits_{(x,y)\to(x_0,y_0)} \varepsilon = 0$ since $\left|\frac{x-x_0}{\sqrt{}}\right|, \left|\frac{y-y_0}{\sqrt{}}\right| < 1$

Similarly

$$\varepsilon \cdot \sqrt{} = \varepsilon \cdot \frac{(x-x_0)^2 + (y-y_0)^2}{\sqrt{}} = \overset{\varepsilon_1 \to 0}{\boxed{\frac{\varepsilon \cdot (x-x_0)}{\sqrt{}}}}(x-x_0) + \overset{\varepsilon_2 \to 0}{\boxed{\frac{\varepsilon \cdot (y-y_0)}{\sqrt{}}}}(y-y_0)$$

Let $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

(a linear function)

Then $f(x,y)$ is differentiable at (x_0,y_0)

$$\Longleftrightarrow \frac{f(x,y)-L(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = \varepsilon \xrightarrow[(x,y)\to(x_0,y_0)]{} 0$$

$\Longleftrightarrow$ $L(x,y)$ is the linearization of $f(x,y)$ at (x_0,y_0)

$\Longleftrightarrow$ $L(x,y)$ is the linear approximation of $f(x,y)$ near (x_0,y_0)

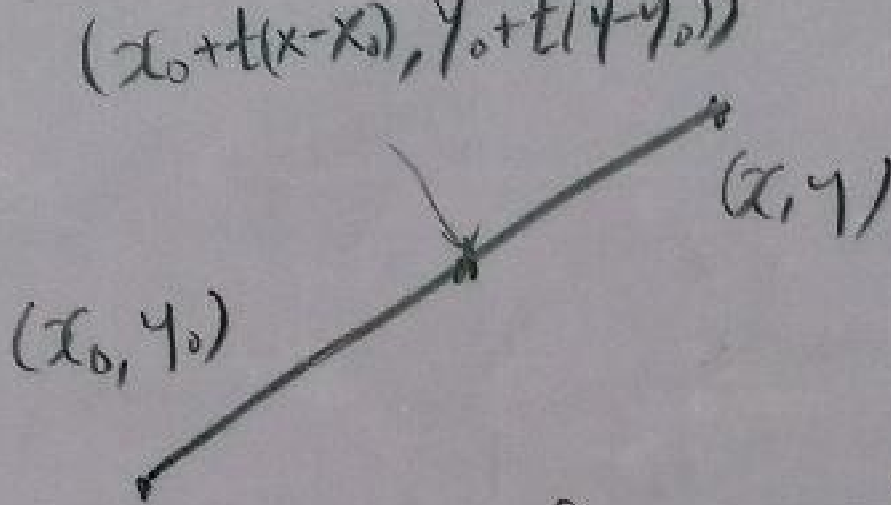
Error of linear approximation

Thm If $f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ are all continuous in an open region R, $(x_0, y_0) \in R$

If $|f_{xx}|, |f_{xy}|, |f_{yy}| \leq M$ on R, Then

$$|f(x,y) - L(x,y)| \leq \frac{M}{2}\left(|x-x_0| + |y-y_0|\right)^2$$

$$\left(\begin{array}{l} \leq M\left((x-x_0)^2 + (y-y_0)^2\right) \\ \because (|a|+|b|)^2 = a^2 + 2|a||b| + b^2 \\ \qquad \leq a^2 + a^2 + b^2 + b^2 \end{array} \right)$$



Let $g(t) = f(\overbrace{x_0+t(x-x_0),\, y_0+t(y-y_0)}^{*})$

$g(0) = f(x_0, y_0)$, $g(1) = f(x, y)$

$$\Rightarrow g(1) = g(0) + g'(0)(1-0) + \frac{g''(c)}{2}(1-0)^2$$

$$g'(t) = f_x(*)(x-x_0) + f_y(*)(y-y_0)$$

$$g''(t) = f_{xx}(*)(x-x_0)^2 + f_{xy}(*)(x-x_0)(y-y_0)$$
$$+ f_{yx}(*)(x-x_0)(y-y_0) + f_{yy}(*)(y-y_0)^2$$

$$\therefore g'(0) = f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)$$

$$|g''(c)| \leq M(x-x_0)^2 + 2M|x-x_0||y-y_0| + M(y-y_0)^2$$

$$= M(|x-x_0| + |y-y_0|)^2$$

$$|g(1) - (g(0) + g'(0))|$$

$$= \left|\frac{g''(c)}{2}\right|$$

$$\therefore |f(x,y) - L(x,y)|$$

$$= \left|\frac{g''(c)}{2}\right| \leq \frac{M}{2}(|x-x_0| + |y-y_0|)^2$$