<u>Def</u> Derivative of $f(x,y)$ at $P_0=(x_0,y_0)$ in the direction of the unit vector $\vec{u}=(u_1,u_2)$ $(u_1^2+u_2^2=1)$:

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} \left(= D_{\vec{u}} f(x_0,y_0)\right)$$

$$= \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$$

$$= \lim_{s\to 0} \frac{f(x(s),y(s)) - f(x(0),y(0))}{s-0}$$

where $x(s)=x_0+su_1$, $y(s)=y_0+su_2$

<u>Theorem</u> If $f(x,y)$ is differentiable at (x_0, y_0)

Then

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

where $\nabla f = (f_x, f_y)$

<u>~~Pf~~</u>

$$\Delta f = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\Delta x = \Delta s\, u_1 \,,\ \Delta y = \Delta s\, u_2$$

$$\Rightarrow \frac{\Delta f}{\Delta s} \xrightarrow[\Delta s \to 0]{} (f_x, f_y) \cdot (u_1, u_2)$$

Example $f(x,y) = x^2 + xy$

$\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (Note if f is differentiable)

$D_{\vec{u}} f(1,2) = ?$

Ans $\lim_{s \to 0} \frac{f(1+\frac{s}{\sqrt{2}}, 2+\frac{s}{\sqrt{2}}) - f(1,2)}{s-0}$

$= \frac{d}{ds} f\left(1+\frac{s}{\sqrt{2}}, 2+\frac{s}{\sqrt{2}}\right)\Big|_{s=0}$

chain rule

$= f_x \cdot \frac{1}{\sqrt{2}} + f_y \cdot \frac{1}{\sqrt{2}} \Big|_{s=0}$

$= (2x+y, x)\Big|_{s=0\ (x=1,\, y=2)} \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$= (4,1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Example $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$$f_x(0,0) = \lim_{s \to 0} \frac{f(s,0) - f(0,0)}{s - 0} = 0$$

$$f_y(0,0) = \lim_{s \to 0} \frac{f(0,s) - f(0,0)}{s - 0} = 0$$

$$D_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} f(0,0) = \lim_{s \to 0} \frac{f\left(\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right) - f(0,0)}{s - 0}$$

$$= \lim_{s \to 0} \frac{\dfrac{s^3 \left(\frac{1}{\sqrt{2}}\right)^3}{s^2}}{s} = \left(\frac{1}{\sqrt{2}}\right)^3$$

$$\neq \nabla f(0,0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 0$$

The Theorem on page 2 says equlity holds, what's wrong?

In contrast, if

$f(x,y) = ax+by+c$ (plane)

$f_x(0,0) = a$, $f_y(0,0) = b$

$$D_{(u_1,u_2)} f(0,0) = \frac{(asu_1+bsu_2+c)-c}{s}$$

$$= (a,b)(u_1,u_2)$$

That is

f has a tangent plane at (x_0,y_0)

($\Leftrightarrow$ f is "close to" a plane)

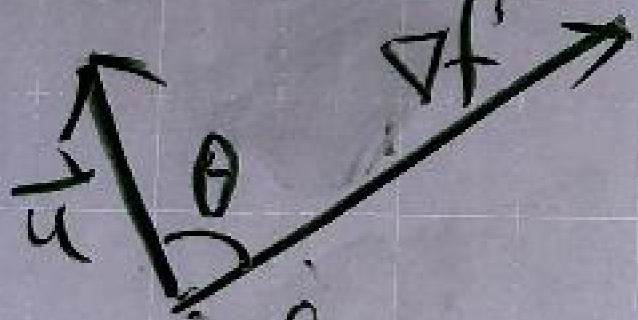
$$\Rightarrow D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \vec{u}$$

Properties of ∇f

If f is differentiable

$$\Rightarrow D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0)\cdot \vec{u}$$

$$= |\nabla f(x_0,y_0)||\vec{u}|\cdot \cos\theta$$

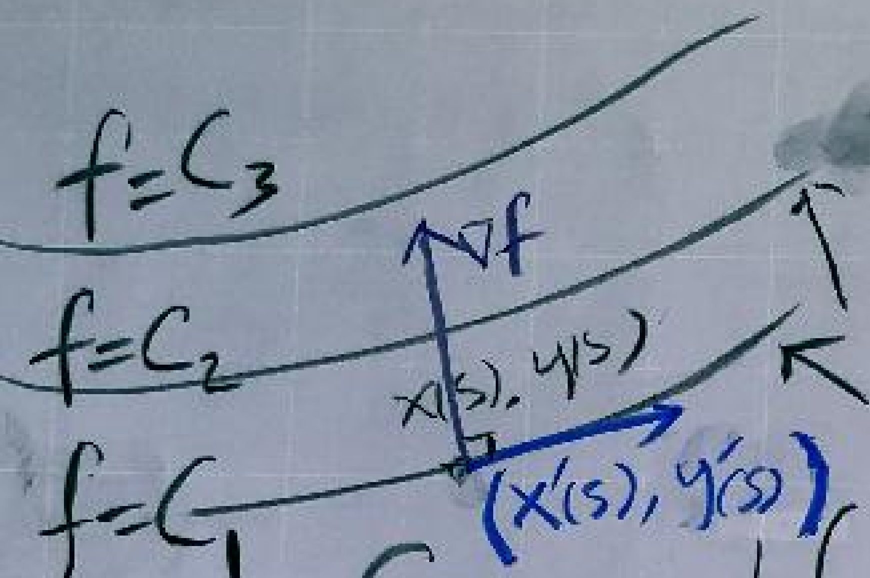


$\Rightarrow$ (1) f increases most rapidly (decreases)

in the direction $\vec{u}$ if $\cos\theta = 1$ ($\cos\theta = -1$),

that is, in the direction of $\nabla f(x_0,y_0)$ ($-\nabla f(x_0,y_0)$)

(2) $D_{\vec{u}} f(x_0,y_0) = 0$ if $\nabla f(x_0,y_0) \perp \vec{u}$



level curves of f

$\nwarrow$ f = constants

$= \{(x,y) \mid f(x,y) = c\}$

If $(x(s), y(s))$ is a level curve of f (differentiable)

$\Rightarrow f(x(s), y(s)) = \text{constant}$ in s

$\frac{d}{ds} \Rightarrow 0 = \nabla f(x(s), y(s)) \cdot \underline{(x'(s), y'(s))}$

direction of tangent line

$\therefore \nabla f \perp (\text{tangent line})$

<u>Example</u> Find the tangent (normal) lines of $\frac{x^2}{4}+y^2=2$ at $(-2,1)$

<u>Sol</u> (a new method)

Let $f(x,y)=\frac{x^2}{4}+y^2$

$\frac{x^2}{4}+y^2=2$ is a level curve of $f(x,y)$

tangent line

$(x+2,\ y-1) \parallel$ tangent vector

$\therefore\ \nabla f(-2,1)\cdot(x+2,\ y-1)=0$

$\Rightarrow -1\cdot(x+2)+2(y-1)=0$

normal line

$(x+2, y-1) \parallel \nabla f(-2,1)$

$(x+2, y-1) \parallel (-1, 2)$

$\therefore -1(y-1) = 2(x+2)$

<u>Rm</u>

(i) $\nabla(f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla(c f(x,y)) = c \nabla f(x,y)$

(iii) $\nabla(fg) = ((fg)_x, (fg)_y)$

$= f\nabla g + g\nabla f$

(iv) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$