Chain Rule

(*) $\frac{d}{dt} f(x(t)) = \frac{df}{dx}(x(t)) \frac{dx(t)}{dt}$

(*) $\frac{d}{dt} f(x(t), y(t))$

$= \frac{\partial f}{\partial x}(x(t), y(t)) \frac{dx(t)}{dt}$

$+ \frac{\partial f}{\partial y}(x(t), y(t)) \frac{dy}{dt}(t)$

$= f_x \cdot x_t + f_y \cdot y_t$

$$(*)\ \frac{\partial f}{\partial s}\left(x(s,t), y(s,t), z(s,t)\right)$$

$$= f_x X_s + f_y \cdot y_s + f_z \cdot Z_s$$

$$\frac{\partial f}{\partial t}(\ldots) = f_x X_t + f_y y_t + f_z Z_t$$

$$(*)\quad \frac{\partial}{\partial s} f\left(x(s,t)\right) = f_x X_s$$

$$\frac{\partial}{\partial t} f\left(x(s,t)\right) = f_x \cdot X_t$$

Implicit differentiation

Suppose $y(x)$ is implicitly defined by $F(x, y) = 0$, then $\frac{dy}{dx} = ?$

<u>Ans.</u> $F(x, y(x)) = 0$

$$\frac{d}{dx} F(x, y(x)) = \frac{\partial F}{\partial x}(x, y(x)) \cdot 1 + \frac{\partial F}{\partial y}(x, y(x)) \cdot \frac{dy}{dx}$$

Example

$$F(x,y,z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9} - 3 = 0$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(x_0, y_0, z_0) = (1, 2, 3)$

Ans $z(x,y)$ is implicitly defined by $F(x, y, z(x,y)) = 0$

$$0 = \frac{\partial F}{\partial x}(x, y, z(x,y)) \text{ (y fixed)}$$

$$= \frac{\partial F}{\partial x}(x, y, z(x,y)) \cdot 1 \text{ (y, z fixed)}$$

$$+ \frac{\partial F}{\partial z}\overset{(x,\,y\text{ fixed})}{(x, y, z(x,y))} \cdot \frac{\partial z}{\partial x}$$

$$\boxed{\therefore \frac{\partial z}{\partial x} = -F_x(x, y, z(x,y)) / F_z(x, y, z(x,y))}$$

$$0 = x^2 + \frac{y^2}{4} + \frac{z^2(x,y)}{9} - 1$$

$$\frac{\partial}{\partial x} \Rightarrow 0 = 2x + \frac{2z}{9}\frac{\partial z}{\partial x}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,2,3)} = -\left. 2x \Big/ \left(\frac{2z}{9}\right)\right|_{(1,2,3)}$$

$$= -2\cdot 1 \Big/ \frac{2\cdot 3}{9} = -3$$

Similarly

$$\boxed{\left.\frac{\partial z}{\partial y}\right|_{(1,2,3)} = -\left. F_y \Big/ F_z \right|_{(1,2,3)}}$$

$$= -\left.\left(\frac{y}{2}\right) \Big/ \left(\frac{2z}{9}\right)\right|_{(1,2,3)} = \frac{-3}{2}$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

Example Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0,0,0)$

if $x^3 + z^2 + ye^{xz} + z\cos y = 0$

Ans:

$$F_x(0,0,0) = 3x^2 + yze^{xz}\Big|_{(0,0,0)} = 0$$

$$F_y(0,0,0) = e^{xz} - z\sin y\Big|_{(0,0,0)} = 1$$

$$F_z(0,0,0) = 2z + xye^{xz} + \cos y = 1$$

$$\frac{\partial z}{\partial x}\Big|_{(0,0,0)} = -\frac{F_x}{F_z}\Big|_{(0,0,0)} = 0$$

$$\frac{\partial z}{\partial y}\Big|_{(0,0,0)} = -\frac{F_y}{F_z}\Big|_{(0,0,0)} = -1$$

Question: What happens if we compute $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ in polar coordinate?

Ans: Can we find L, such that

$$\lim_{r\to 0} \max_{0<x^2+y^2\leq r^2} |f(x,y)-L| \overset{?}{=} 0$$

$$\lim_{\substack{r\to 0 \\ \theta \text{ fixed}}} \frac{r^3\cos\theta\sin^2\theta}{r^2\cos^2\theta + r^4\sin^4\theta} = 0$$

The 1d lim = 0 along every straight line passing through 0, but the 2d lim need not = 0

$$\lim_{r\to 0}\ \max_{\substack{\theta\in[0,2\pi]\\ 0<\rho\le r}} \left|\frac{\rho\cos\theta\sin^2\theta}{\cos^2\theta+\rho^2\sin^4\theta}\right|$$

$$\overset{\rho=r}{\geq} \lim_{r\to 0}\ \max_{\theta\in[0,2\pi]} \frac{r\cos\theta\sin^2\theta}{\cos^2\theta+r^2\sin^4\theta}$$

Fix k. take $\theta=\theta(r)$ such that $\begin{array}{l}\cos\theta=kr\\ \sin^2\theta=1-k^2r^2\end{array}$

$$\geq \lim_{r\to 0}\frac{r\cdot kr(1-k^2r^2)}{k^2r^2+r^2(1-k^2r^2)^2}=\lim_{r\to 0}\frac{kr^2}{k^2r^2+r^2}=\frac{k}{1+k^2}$$

$\therefore$ 2d $\lim\neq 0$ (it does not exist)