Def: $f(x,y)$ is differentiable at (x_0,y_0) if

$$f(x,y) = \boxed{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

$$+ \varepsilon_1(x-x_0) + \varepsilon_2(y-y_0)$$

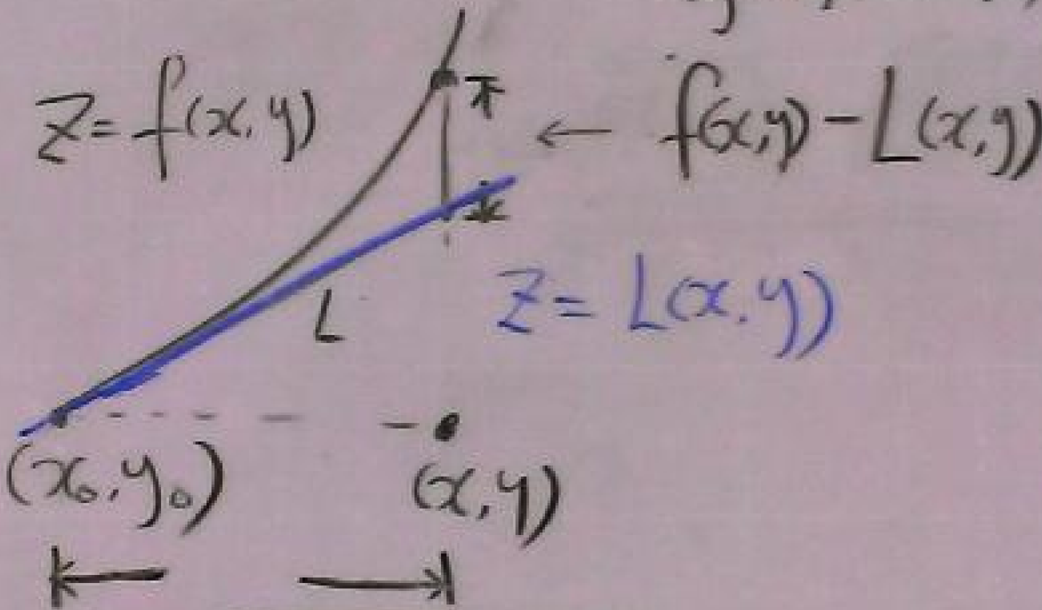
$$\left(\text{or } + \varepsilon\cdot\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$$

$$\left[\text{or } \underset{\substack{\downarrow \\ f(x,y)-f(x_0,y_0)}}{\Delta z} = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \underset{\substack{\downarrow \\ x-x_0}}{\Delta x} + \varepsilon_2 \underset{\substack{\downarrow \\ y-y_0}}{\Delta y} \ (\text{or } \ )\right]$$

where $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \varepsilon, \varepsilon_1, \varepsilon_2 = 0$

<u>Rm</u> $f(x,y)$ and

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

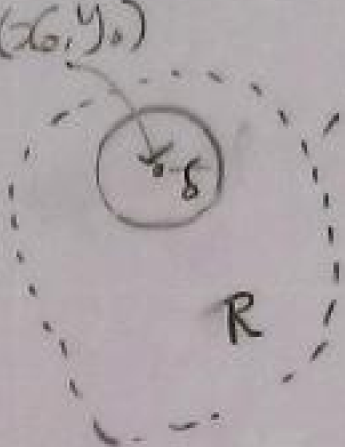


$$d = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

Differentiable at (x_0,y_0)

$$\Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = \lim \varepsilon = 0$$

Def R is an open region if $(x_0, y_0) \in R$



then there exists a corresponding $\delta > 0$, such that

$$\left\{ (x,y) \,\middle|\, \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \right\} \subseteq R$$

Example $\{x^2 + y^2 < 1\}$ is an open region.

$\{x^2 + y^2 \leq 1\}$ is NOT open

When is f differentiable at (x_0, y_0)?

Note: $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ exist

$\not\Longrightarrow$ f is diff. at (x_0, y_0).

(*) **Thm** R: open region.
$(x_0, y_0) \in R$. If f is defined on R, f_x, f_y are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0)

Example $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

(i) f is continuous in $\mathbb{R}^2$

(ii) f_x, f_y are cont. in $\mathbb{R}^2 \setminus (0,0)$

(iii) $f_x = \dfrac{y}{\sqrt{x^2+y^2}} - \dfrac{1}{2}\dfrac{2x^2y}{\sqrt{x^2+y^2}^3} = \dfrac{y^3}{\sqrt{x^2+y^2}^3}$

$(x,y) \neq (0,0)$

$f_x(0,0) = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = 0$

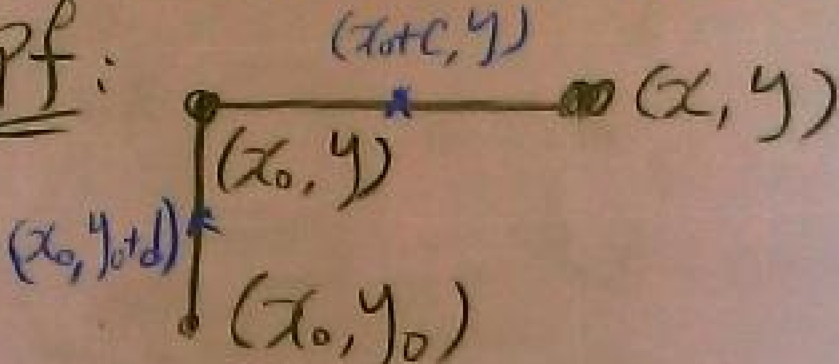
$\therefore f_x$ is not cont. at $(0,0)$

Similarly for f_y

(iv) We will show later that $f(x,y)$ is not differentiable at $(0,0)$.

(Proof of the Theorem (*) on page 4)

Pf:



$$f(x,y)-f(x_0,y_0)$$

$$=\big(f(x,y)-f(x_0,y)\big)+\big(f(x_0,y)-f(x_0,y_0)\big)$$

$$= f_x(\underline{x_0+c},y)(x-x_0)+f_y(x_0,\underline{y_0+d})(y-y_0)$$

$$= f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

$$+\underbrace{\begin{pmatrix} f_x(\underline{x_0+c},y) \\ -f_x(x_0,y_0)\end{pmatrix}}_{\varepsilon_1}(x-x_0)+\underbrace{\begin{pmatrix} f_y(x_0,\underline{y_0+d}) \\ -f_y(x_0,y_0)\end{pmatrix}}_{\varepsilon_2}(y-y_0)$$

__Corollary__: If f is diff. at (x_0, y_0)
then f is cont. at (x_0, y_0)

__Pf__ f is diff at (x_0, y_0)

$$\Rightarrow \Delta z = \boxed{f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

$$\therefore \lim_{(\Delta x,\Delta y)\to(0,0)} \Delta z = \lim_{(\Delta x,\Delta y)\to(0,0)} \boxed{\;} = 0$$

$$\parallel$$

$$\therefore \lim_{(x,y)\to(x_0,y_0)} f(x,y) - f(x_0,y_0)$$

$$\therefore \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0) \quad \left(\begin{array}{c} f \text{ is cont} \\ \text{at } (x_0,y_0) \end{array}\right)$$

Thm If $f, f_x, f_y, f_{xy}, f_{yx}$ are cont. in an open region R and $(x_0, y_0) \in R$

Then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$

where $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$.

Remark $f(x,y) = \begin{cases} xy\,\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Then $f_{xy} = f_{yx}$ on $\mathbb{R}^2 \setminus (0,0)$

and $f_{xy}(0,0) \neq f_{yx}(0,0)$

The Chain Rule

Single variable:

$$\frac{d}{dx} f(g(x)) = \frac{df}{dy}\Big|_{y=g(x)} \cdot \frac{dg(x)}{dx}$$

Multi-variable

$$\frac{d}{dt} f(x(t), y(t))\Big|_{t=t_0} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$$

$$= f_x(x(t_0), y(t_0))\, x'(t_0) + f_y(x(t_0), y(t_0))\, y'(t_0)$$

or

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t)) \frac{dx}{dt} + f_y(x(t), y(t)) \frac{dy}{dt}$$

Similarly,

$$\frac{\partial}{\partial s} f(x(s,t), y(s,t), z(s,t))$$

$$= f_x(x(s,t), y(s,t), z(s,t)) \cdot x_s(s,t)$$
$$+ f_y(\quad " \quad) \cdot y_s(s,t)$$
$$+ f_z(\quad " \quad) \cdot z_s(s,t)$$

and Similarly for

$$\frac{\partial}{\partial t} f(x(s,t), y(s,t), z(s,t))$$