Continuity of $f(x,y)$

<u>Def</u> $f(x,y)$ is continuous at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

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i.e. Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Longrightarrow |f(x,y)-f(x_0,y_0)| < \varepsilon$$

<u>Example</u> $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Then (i) $f(x,y)$ is continuous at $(x,y) \neq (0,0)$.

(ii) Is $f(x,y)$ cont. at $(0,0)$?

<u>Ans</u> No:

$$\frac{2xy}{x^2+y^2} = 2\cos\theta \cdot \sin\theta$$

$\lim_{r \to 0}$ is different for diffrent θ.

$\therefore$ $f(x,y)$ is cont. on $\mathbb{R}^2 \setminus (0,0)$

Example $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = ?$

Let $y = mx$, $x \to 0$

$$\lim_{x\to 0} \frac{2x^2mx}{x^4+m^2x^2} = 0$$

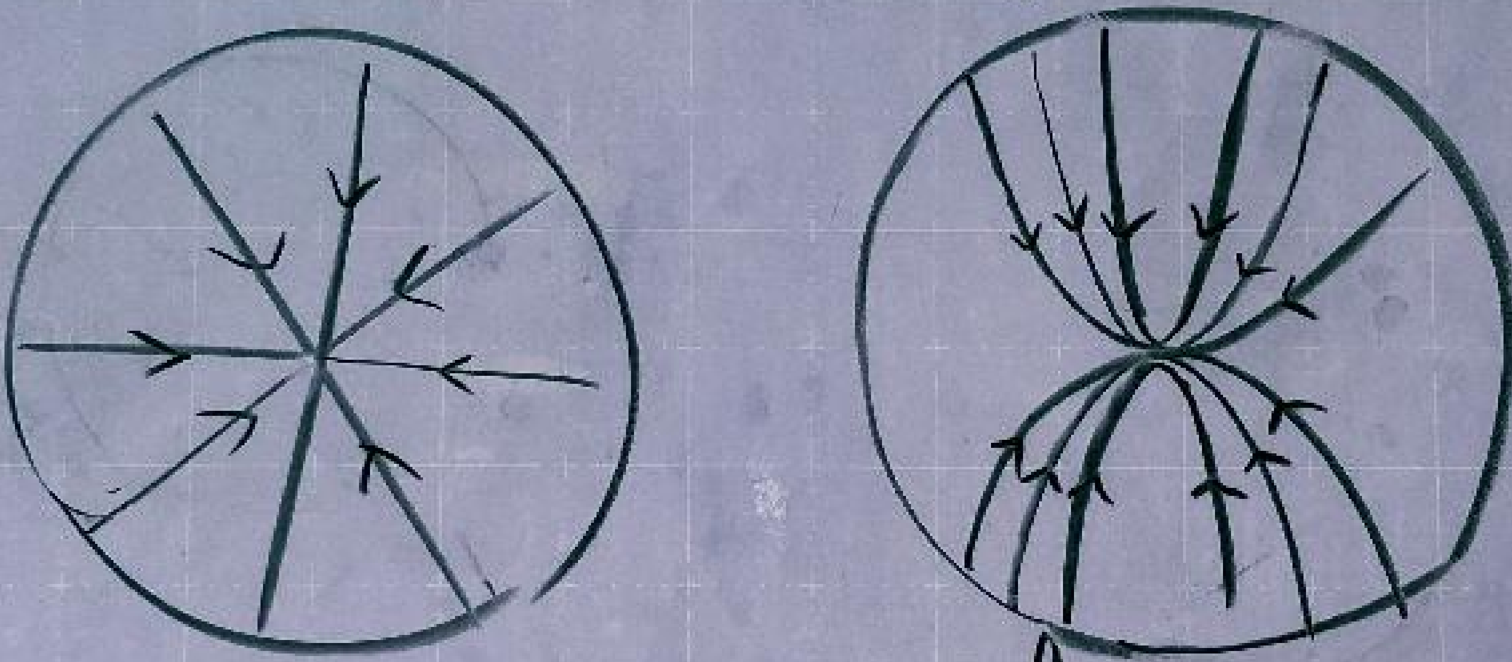
What if $y = kx^2$, $x \to 0$?

$$\lim_{x\to 0} \frac{2kx^4}{x^4(1+k^2)} = \frac{2k}{1+k^2}$$

In other words

$f(x,y) \to 0$ when

$(x,y) \to (0,0)$ along straight lines



But $f(x,y) \to \dfrac{2k}{1+k^2}$ when

$(x,y) \to (0,0)$ along $y = kx^2$

From "two path test"

limit does not exist.

es

Example Continuous extension

Exa

$$f(x,y) = xy\,\frac{x^2-y^2}{x^2+y^2}$$

$(x$

f is defined and cont. on $\mathbb{R}^2 \setminus (0,0)$. Is it possible to define $f(0,0)$ so that f is cont. in $\mathbb{R}^2$?

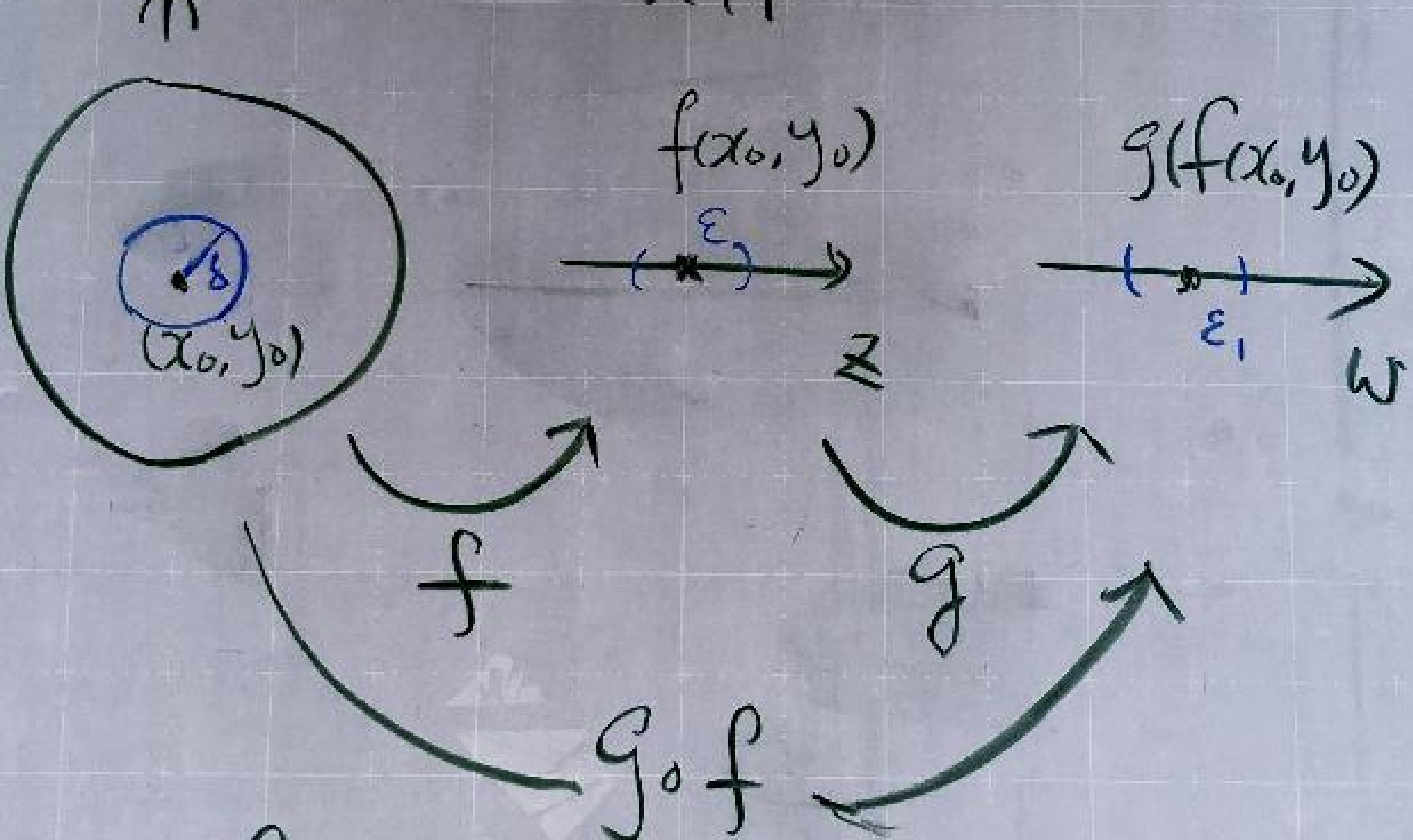
$$\lim_{(x,y)\to(0,0)} xy\,\frac{x^2-y^2}{x^2+y^2} = \lim_{r\to 0} r^2 \cos\theta \sin\theta \cdot (\cos^2\theta - \sin^2\theta)$$

$$= 0$$

Ans. Define $f(0,0) = 0$

Example $\cos\left(\frac{xy}{x^2+1}\right)$

$$(x,y) \xrightarrow{f} z = \frac{xy}{x^2+1} \xrightarrow{g} w = \cos z$$



If $\begin{cases} f \text{ is cont. at } (x_0,y_0) \\ g \text{ is cont. at } f(x_0,y_0) \end{cases}$

Then $g \circ f$ is cont. at (x_0,y_0)

g is cont. at $f(x_0, y_0)$

$$\begin{cases} \text{Given } \varepsilon_1 > 0, \ \exists\, \varepsilon > 0, \text{ so that} \\ |z - f(x_0, y_0)| < \varepsilon \Rightarrow |g(z) - g(f(x_0, y_0))| < \varepsilon_1 \end{cases} \quad (*)$$

f is cont. at (x_0, y_0)

$$\begin{cases} \text{Given } \varepsilon > 0, \ \exists\, \delta > 0 \text{ so that} \\ \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \Longrightarrow |f(x,y) - f(x_0, y_0)| < \varepsilon \end{cases} \quad (**)$$

For $g \circ f$, given $\varepsilon_1 > 0$, find $\varepsilon > 0$ from $(*)$

for this $\varepsilon > 0$, find $\delta > 0$ from $(**)$

Then given $\varepsilon > 0$

$$\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

$$\overset{(**)}{\Longrightarrow} \left| f(x,y) - f(x_0,y_0) \right| < \varepsilon$$

$$\downarrow$$

$$z$$

$$\overset{(*)}{\Longrightarrow} \left| g(f(x,y)) - g(f(x_0,y_0)) \right| < \varepsilon_1$$

$\Rightarrow g \circ f$ is cont. at (x_0, y_0)

Continuity of composite functions

Partial Derivatives

<u>**Def**</u>

$$\begin{cases} \dfrac{\partial f}{\partial x}(x_0, y_0) = \lim\limits_{h \to 0} \dfrac{f(x_0+h, y_0) - f(x_0, y_0)}{h} \\[2ex] \dfrac{\partial f}{\partial y}(x_0, y_0) = \lim\limits_{k \to 0} \dfrac{f(x_0, y_0+k) - f(x_0, y_0)}{k} \end{cases}$$

Notation $\dfrac{\partial f}{\partial x} = f_x = \partial_x f$

$$\frac{\partial f}{\partial y} = f_y = \partial_y f$$

Example $f(x,y) = x^2 + 3xy + y - 1$

$\frac{\partial f}{\partial x} = ?$, $\frac{\partial f}{\partial y} = ?$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(x,y)$$

$$= 2x + 3y + 0$$

treat y as a constant

$$\frac{\partial f}{\partial y}(x,y) = 0 + 3x + 1$$

$x \simeq$ constant

ry **Example** Find $\frac{\partial z}{\partial x}$ if

$$yz - \ln z = x + y$$

(That means $z(x,y)$ is implicitly defined by
$yz(x,y) - \ln z(x,y) = x + y$)

$$\frac{\partial}{\partial x} \Longrightarrow y z_x - \frac{z_x}{z} = 1$$

(y=constant)

$$\therefore \frac{\partial z}{\partial x}(x,y) = \frac{1}{y - \frac{1}{z(x,y)}} = \frac{1}{y - \frac{1}{z}}$$

Example $f(x,y,z) = x\sin(y+3z)$

$$\frac{\partial f}{\partial x} \underset{\substack{y=\text{const}\\ z=\text{const}}}{=} \sin(y+3z)$$

$$\frac{\partial f}{\partial y} \underset{\substack{x=\text{const}\\ z=\text{const}}}{=} x\cos(y+3z)$$

$$\frac{\partial f}{\partial z} \underset{\substack{x=\text{const}\\ y=\text{const}}}{=} x\left(\cos(y+3z)\right)\cdot 3$$

1D	2D
$f(x)$ is cont at x_0	$f(x,y)$ is cont at (x_0,y_0)
$\frac{df}{dx}(x_0)$ exists	$\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$ exist
$\Leftrightarrow$ f is diff. at x_0	$\Leftarrow$ / $\not\Leftrightarrow$ f is diff. at (x_0,y_0)
$\Leftrightarrow$ f can be approximated by $y=f(x_0)+f'(x_0)(x-x_0)$ near $(x_0, f(x_0))$	$\Leftrightarrow$ $f(x,y)$ can be approximated by $z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$ near $(x_0,y_0,f(x_0,y_0))$
(Existence of tangent line)	(Existence of tangent plane)

Ex

Example

$$f(x,y)=\begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$

(i) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\because \lim_{\substack{x\to 0 \\ y=0}} f(x,y) = 1$, $\lim_{\substack{x=y \\ x\to 0}} f(x,y) = 0$

(ii) $f(x,y)$ is not cont. at $(0,0)$

(iii) $\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = 0$

$\frac{\partial f}{\partial x}(x_0,y_0)$, $\frac{\partial f}{\partial y}(x_0,y_0)$ exist $\not\Longrightarrow$ f is continuous at (x_0,y_0)

Def $z = f(x,y)$ is differentiable at (x_0, y_0) if $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ exist and

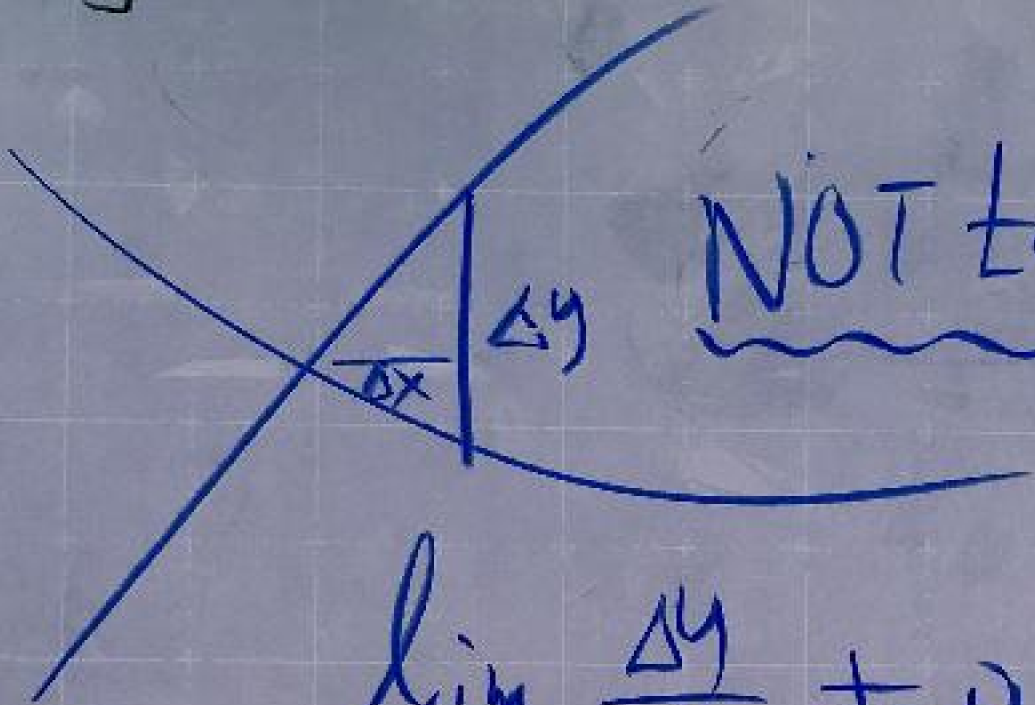
$$f(x,y) = \boxed{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

$$+ \underbrace{\varepsilon_1 (x-x_0) + \varepsilon_2 (y-y_0)}$$

$$\left(\text{or } + \varepsilon \cdot \sqrt{(x-x_0)^2 + (y-y_0)^2} \right)$$

where $\lim\limits_{(x,y)\to(x_0,y_0)} \varepsilon, \varepsilon_1, \varepsilon_2 = 0$

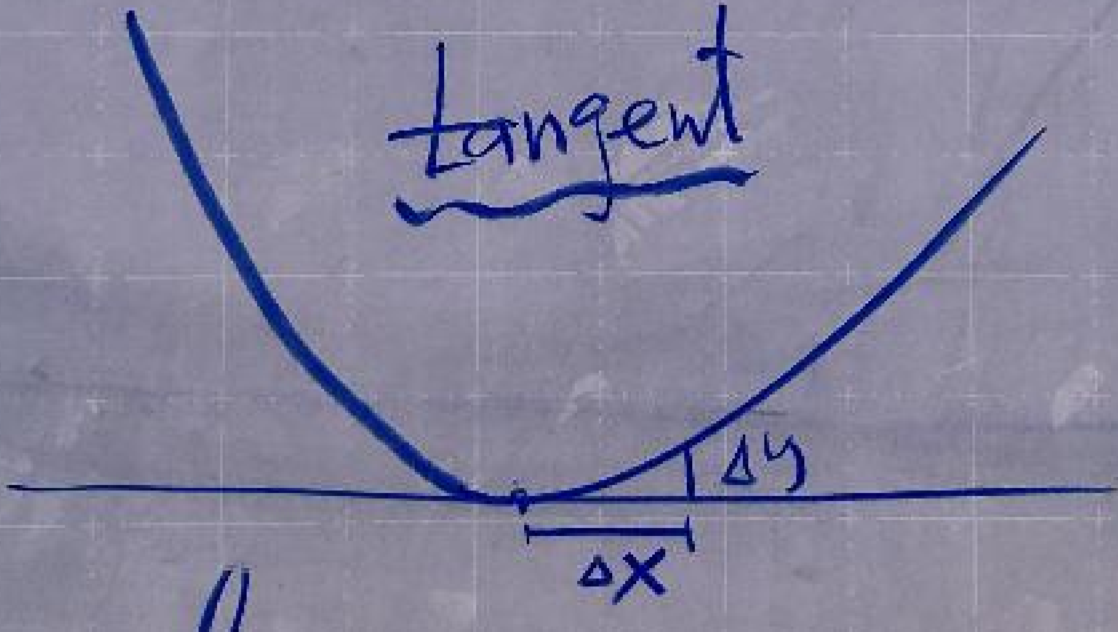
Tangent Vs Not tangent



NOT tangent

$$\lim \frac{\Delta y}{\Delta x} \neq 0$$

tangent



$$\lim \frac{\Delta y}{\Delta x} = 0$$