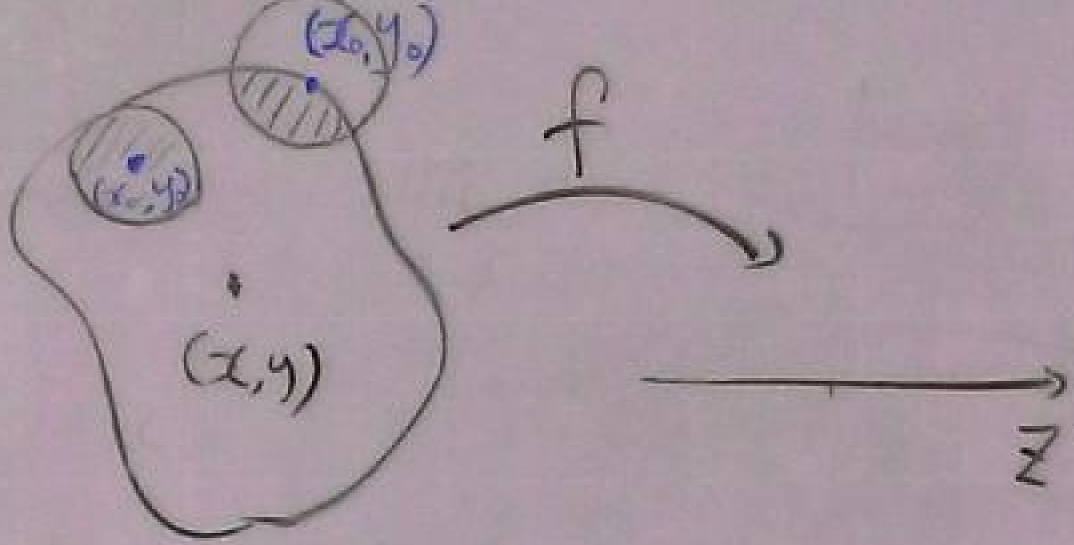
Limit and Continuity in higher dimensions:

2D case



$$f : (\text{domain of } f) \longmapsto \mathbb{R}$$

$$(x, y) \longmapsto z = f(x, y)$$

__Def__ $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

if for every $\varepsilon > 0$

there exists a corresponding

$\delta > 0$, such that for

all (x,y) in the domain

of f

$|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

(i.e. $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \Longrightarrow |f(x,y) - L| < \varepsilon$)

Example $\lim\limits_{(x,y)\to(x_0,y_0)} x = x_0$

Pf for any $\varepsilon > 0$

find $\delta > 0$, such that

$|x - x_0| < \varepsilon$ whenever

$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

Take $\delta = \varepsilon$,

Verification:

$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \varepsilon$

$\Rightarrow |x - x_0| < \sqrt{(\ )} < \varepsilon$

Example $\lim\limits_{(x,y)\to(0,1)} \dfrac{x - xy + 3}{x + y^2}$

$$= \frac{(\lim x) - (\lim x)(\lim y) + 3}{(\lim x) + (\lim y)^2}$$

$$= \frac{0 - 0 \cdot 1 + 3}{0 + 1^2}$$

$$= 3$$

Example $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Remark domain $= \left\{ \begin{array}{l} x \geq 0,\ y \geq 0 \\ xy \neq 0 \end{array} \right\}$

note that $(0,0) \notin$ domain

For $(x,y) \in$ domain

$$f(x,y) = \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = x(\sqrt{x}+\sqrt{y})$$

$$\boxed{\lim_{(x,y)\to(0,0)} f(x,y) = 0(0+0) = 0}$$

Example $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$

In polar coordinate

$0 < r < \delta$

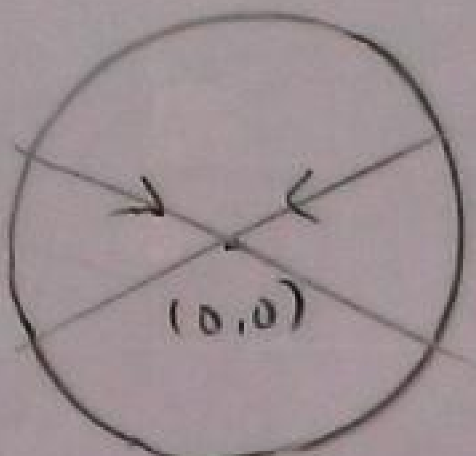
$\overset{?}{\Rightarrow} \left| \underset{f(x,y)}{\underset{\shortparallel}{f(r\cos\theta, r\sin\theta)}} - L \right| < \varepsilon$

$f(x,y) = 4r\cos\theta\sin^2\theta \Rightarrow \boxed{L=0}$

For any $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{4}$

$0 < \sqrt{} < \delta$
$(0 < r < \delta)$ $\Rightarrow |f(x,y) - 0| \le 4r < 4\delta = \varepsilon$

Example $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x} = ?$



Let $(x,y)\to(0,0)$ along the path $y = mx$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{y}{x} = \lim_{x\to 0} \frac{mx}{x} = m$$

$\therefore$ different limit along different m on path $y=mx$

$\therefore$ limit does not exist.

Example $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2} = ?$

Ans: Let $(x,y)\to(0,0)$
along path $y=mx$
$f(x,y)=\dfrac{2m}{1+m^2}$ gives
different value for
different m, no
matter how small
$\sqrt{(x-0)^2+(y-0)^2}$ is

Two path Thm $\Rightarrow$ limit does not exist

Two path theorem:

If $f(x,y)$ has different limits along two paths in the domain of f as $(x,y) \longrightarrow (x_0, y_0)$,

Then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist