Summary of known Taylor Series

Memorize the following:

$$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \cdots ; |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad x \in \mathbb{R}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \quad x \in \mathbb{R}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots; \quad -1 < x \leq 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \quad |x| < 1$$

Remark. Formally,

we can write

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

In fact, "=" holds for all $\theta \in \mathbb{R}$ (beyond this course)

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)$$

$$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

Application:

$$\int e^{ax} \cos bx \, dx = ?$$

Method 1: integration by parts twice.

Method 2:

$$= \mathrm{Re} \int e^{ax} e^{ibx} dx$$

Re means real part of

$$= \mathrm{Re} \int e^{(a+ib)x} dx$$

$$= \mathrm{Re} \left(\frac{1}{a+ib} e^{(a+ib)x} + C \right)$$

$= \cdots$ homework

Rm

<u>Rm</u>: $\ln(1+1) \stackrel{?}{=} 1 - \frac{1}{2} + \frac{1}{3} - \cdots$

$$\ln(1+x)' = \frac{1}{1+x}$$

$$\ln(1+x)^{(n+1)} = \left(\frac{1}{1+x}\right)^{(n)}$$

$$\left(\frac{1}{1+x}\right)' = -(1+x)^{-2}$$

$$\left(\frac{1}{1+x}\right)'' = 2(1+x)^{-3}$$

$$\left(\frac{1}{1+x}\right)''' = -3!(1+x)^{-4}$$

$$\left(\frac{1}{1+x}\right)^{(n)} = (-1)^n n! (1+x)^{-n}$$

$$= \ln(1+x)^{(n+1)}$$

$$R_n(1) = \frac{(-1)^n n! (1+C_{n+1})^{-n}}{(n+1)!}$$

Note $x=1$, $0 < C_{n+1} < 1$

$\therefore (1+C_{n+1})^{-n} \leq 1$

$\therefore R_n(1) \to 0$ as $n \to \infty$

"=" holds

Examples $\lim\limits_{x\to 0} \dfrac{\sin x-\tan x}{x^3}$

Method 1: l'Hôpital's Rule

Method 2:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5$$

$$\Big(= \underbrace{\left(1 \;\; 0 \;\; \tfrac{-1}{2} \cdots\right)}_{\cos x} \; \begin{array}{r|rrrrl} & & 1 & 0 & \tfrac{1}{3} & \\ \hline & 0 & 1 & 0 & \tfrac{-1}{6} & \cdots \\ & & 1 & 0 & \tfrac{-1}{2} & \\ \hline & & 0 & 0 & \tfrac{1}{3} & \end{array}$$

$$\therefore \text{Ans} = -\frac{1}{6} - \frac{1}{3} = \frac{-1}{2}$$

Example $\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$

$$= \lim_{x\to 0} \frac{x - \sin x}{x^2 \sin x}$$

(l'Hôpital: OK)

$$= \lim_{x\to 0} \frac{x - x + \frac{x^3}{3!} - \cdots}{x^2\left(x - \frac{x^3}{3!} + \cdots\right)}$$

$$= \frac{1}{6}$$

Eg $\frac{1}{1\cdot 2^1} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots$;

(Convergence: Leibnitz test)

$$= \ln\left(1+\frac{1}{2}\right) = \ln\left(\frac{3}{2}\right)$$

Eg $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots \quad |x|<1$

Method 1;

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \cdots\cdots$$

$$\text{Ans} = \begin{cases} \frac{-\ln(1-x)}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$$

Method 2:

$$\left(x\left(1+\frac{x}{2}+\frac{x^2}{3}+\cdots\right)\right)'$$

$$=\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)'$$

$$=1+x+x^2+\cdots=\frac{1}{1-x}$$

$$\therefore x\left(1+\frac{x}{2}+\cdots\right)=\int_0^x\frac{1}{1-t}\,dt$$

$$=-\ln(1-t)\Big|_0^x=-\ln(1-x)$$

$$\therefore Ans=\begin{cases}-\dfrac{\ln(1-x)}{x}, & x\neq 0\\ 1, & x=0\end{cases}$$

Eg $x + 2x^2 + 3x^3 + 4x^4 + \cdots = ?\,, \quad |x| < 1$

Sol $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$$\left(\frac{1}{1-x}\right)' = 1 + 2x + 3x^2 + \cdots$$

Ans $= x\left(\dfrac{1}{1-x}\right)' = \dfrac{-x}{1-x^2}$

Eg $|x|<1$, Evaluate

$$x+2^2x^2+3^2x^3+4^2x^4+\cdots$$

Sol $=\displaystyle\sum_{n=1}^{\infty} n^2x^n$

$$\frac{1}{1-x}=1+\sum_{n=1}^{\infty}x^n$$

$$\left(\frac{1}{1-x}\right)'=\sum_{n=1}^{\infty}nx^{n-1}$$

$$\left(\frac{1}{1-x}\right)''=\sum_{n=1}^{\infty}n(n-1)x^{n-2}$$

$$=\sum_{n=1}^{\infty}(n^2-n)x^{n-2}$$

$$x\left(\frac{1}{1-x}\right)' = \sum_{n=1}^{\infty} n x^n$$

$$x^2\left(\frac{1}{1-x}\right)'' = \sum_{n=1}^{\infty} (n^2-n) x^n$$

$+)$

$$\underbrace{x\left(\frac{1}{1-x}\right)' + x^2\left(\frac{1}{1-x}\right)''}_{\text{Ans}} = \sum_{n=1}^{\infty} n^2 x^n$$

Eg Estimate the error of approximating e^x by $1+x+\frac{x^2}{2!}$ on $|x|<0.1$

Ans. $e^x = P_2(x) + R_2(x)$

$$R_2(x) = \frac{f'''(c_3)}{3!} x^3$$

where $0 < c_3 < x$ if $x > 0$
$x < c_3 < 0$ if $x < 0$

$$|R_2(x)| \leq \frac{e^{c_3}}{3!} |x|^3 \leq \frac{e^{0.1}}{6} 10^{-3}$$