Binomial Series

$\text{Taylor}((1+x)^m, 0)(x) = ?$

(1) $m \in \mathbb{N} \Rightarrow \text{Ans} = (1+x)^m$
(Polynomial)

(2) $m \notin \mathbb{N}$

$f(x) = (1+x)^m$

$f'(x) = m(1+x)^{m-1}$

$f''(x) = m(m-1)(1+x)^{m-2}$

$\cdots$

$\because (1+x)^m = P_n(x) + R_n(x)$

$$P_n(x) = \sum_{k=0}^{n} \binom{m}{k} x^k$$

where

$$\binom{m}{k} = m(m-1)\cdots(m-k+1)/k!$$

$$R_n(x) = \binom{m}{n+1}(1+c_{n+1})^{m-n-1} x^{n+1}$$

Let $n \to \infty$, $m-n-1<0$

$(1+c_{n+1})^{m-n-1} \not\leq M$ for all n

* <u>Not sure</u> if $c_{n+1}<0$ $(x<0)$

It can be shown (indirectly) that $\lim_{n\to\infty} R_n(x) = 0$ on $|x|<1$.

(Homework)

$\therefore (1+x)^m = \text{Taylor}\left((1+x)^m, 0\right)(x)$

on $|x|<1$. $\forall m \in \mathbb{R}$

Example: $\text{Taylor}(\tan x, 0)(x) = ?$

$\tan x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right)^{-1}$

$y = \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \quad (1-y)^{-1} = 1 + y + y^2 + \cdots$

$$= x\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)\cdot\left(1+\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\cdots\right)+\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\cdots\right)^2+\cdots\right)$$

upto x^5

$$= x\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}\right)\left(1+\frac{x^2}{2!}-\frac{x^4}{4!}+\left(\frac{x^2}{2!}\right)^2\right)+\cdots$$

$$= x\left(1-\frac{x^2}{6}+\frac{x^4}{120}\right)\left(1+\frac{x^2}{2}+\frac{5}{24}x^4\right)+\cdots$$

$$= x+\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots\cdots$$

Example $\frac{1}{\sin x}$, $0<|x|<1$

$$= \frac{1}{x\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)}$$

$$= \frac{1}{x}\left(1-\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\cdots\right)\right)^{-1}$$

$$= \frac{1}{x}\left(1+\left(\frac{x^2}{3!}\right)+\cdots\right)$$

$$= \frac{1}{x}+\frac{x}{6}+\cdots$$

$$= \frac{1}{x} + \text{Power series}$$

Example: $\text{Taylor}(\sin^{-1}x, 0)(x) = ?$
(first few terms)

$$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \sin^{-1}x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$$

$$= \int_0^x (1-t^2)^{-\frac{1}{2}}\,dt$$

$$(1+y)^{-\frac{1}{2}} = 1 + \left(\frac{-1}{2}\right)y + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}y^2 + \cdots$$

(let $y = -t^2$)

$$= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots\right)dt$$

(valid for all $|y| = t^2 < 1$)

(Term by term integration)

$$= t + \frac{t^3}{6} + \frac{3}{40}t^5 + \cdots \Big|_0^x$$

$$= x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots, \quad |x| < 1$$