Convergence of Taylor Series

$f(x) \to \mathrm{Taylor}(f, a)(x)$

Does $f(x) \overset{?}{=} \mathrm{Taylor}(f, a)(x)$

Eg: $f(x) = \begin{cases} 0 & x = 0 \\ e^{\frac{-1}{x^2}}, & x \neq 0 \end{cases}$

$f(x) \neq \mathrm{Taylor}(f, 0)(x)$

if $x \neq 0$

Taylor formula

If $f, f', \dots, f^{(n)}, \dots$

all exist on $(a-\delta, a+\delta)$.

Then for any $n \in \mathbb{N}$, $|x-a| < \delta$

$$f(x) = P_n(x) + R_n(x)$$

where $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x

$$f(x) = P_n(x) + R_n(x)$$

Note: $\mathrm{Taylor}(f, a)(x) = \lim_{n\to\infty} P_n(x)$

$\therefore \mathrm{Taylor}(f, a)(x) = f(x)$

$\Longleftrightarrow \lim_{n\to\infty} R_n(x) = 0$

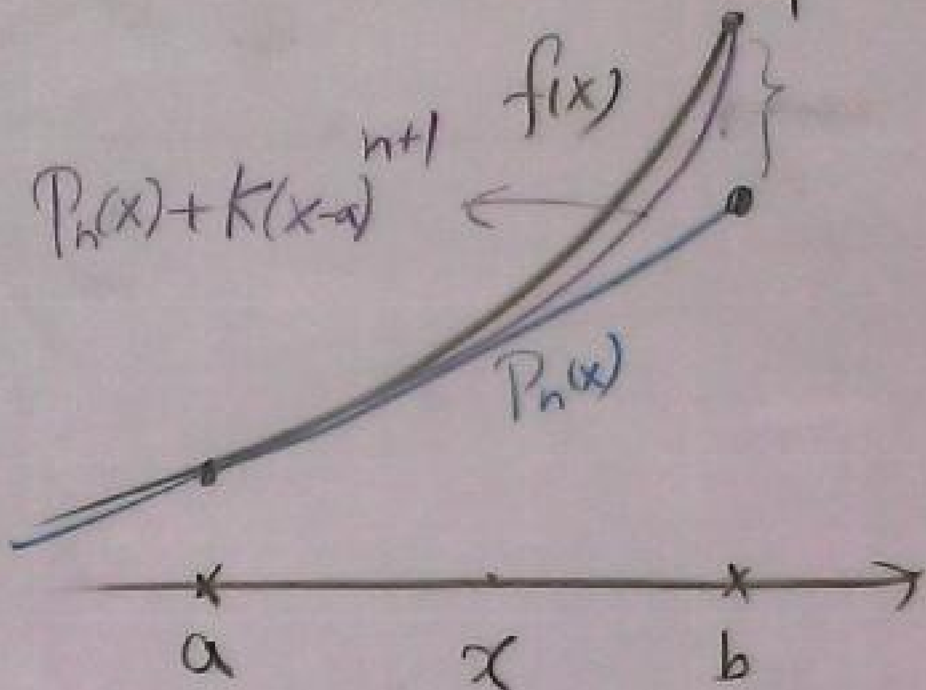
Pf of Taylor formula

We will show that

$$f(b) = P_n(b) + R_n(b)$$

(assume $a < b < a+\delta$)

$$P_n(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$



Find $K \in \mathbb{R}$ such that

$$\boxed{f(b) = P_n(b) + K(b-a)^{n+1}} \quad (*)$$

Then consider

$$F(x) = f(x) - \left(P_n(x) + K(x-a)^{n+1}\right)$$

$$F(a)=0=F(b) \Rightarrow F'(c_1)=0, \quad c_1 \in (a,b)$$

$$F'(a)=f'(a)-\left(P_n'(a)+K(n+1)(a-a)^n\right)=0$$

$$F'(a)=0=F'(c_1) \Rightarrow F''(c_2)=0, \quad c_2 \in (a,c_1)$$

$$F''(a)=f''(a)-\left(P_n''(a)+Kn(n+1)(a-a)^{n-1}\right)$$

$$F''(a)=0=F''(c_2) \Rightarrow F'''(c_3)=0, \quad c_3 \in (a,c_2)$$

$$\vdots$$

$$\Rightarrow F^{(n)}(c_n)=0, \quad c_n \in (a, c_{n-1})$$

$$F^{(n)}(a)=0=F^{(n)}(c_n)$$

$$\Rightarrow F^{(n+1)}(c)=0, \quad c=c_{n+1} \in (a,c_n) \subseteq (a,b)$$

$$0 = F^{(n+1)}(c) = f^{(n+1)}(c) - \left(\underset{\substack{\shortparallel \\ 0}}{P_n^{(n+1)}(c)} - k(n+1)!\right)$$

$$\Rightarrow k = \frac{f^{(n+1)}(c)}{(n+1)!}$$

$$(*) \Rightarrow f(b) = P_n(b) + \overbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}}^{R_n(b)}$$

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<u>Example</u>. $f(x) = e^x$. Taylor$(e^x, 0) \overset{?}{=} e^x$

$f^{(n)}(0) = 1$, $\therefore P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$

$R_n(x) = \dfrac{e^c x^{n+1}}{(n+1)!}$, here $c = c_{n+1}$

$$e^c \le \begin{cases} 1 & \text{if } x < c < 0 \\ e^x & \text{if } 0 < c < x \end{cases}$$

$$\lim_{n\to\infty}|R_n(x)| \le \max(1,e^x)\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0$$

$$\therefore\ e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \text{ for all } x\in\mathbb{R}.$$

Example. $\mathrm{Taylor}(\cos x, 0)(x) = \cos x$

$$\because\ \begin{array}{ccccc} f(0), & f'(0), & f''(0), & f'''(0), & f^{(4)}(0), \cdots \\ \| & \| & \| & \| & \| \\ \cos 0, & -\sin 0, & -\cos 0, & \sin 0, & \cos 0, \cdots \\ 1, & 0, & -1, & 0, & 1, \cdots \end{array}$$

$$P_{2k}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k\frac{x^{2k}}{(2k)!}$$

$$|R_{2k}(x)| = \frac{|\sin c|}{(2k+1)!}|x|^{2k+1} \xrightarrow[k\to\infty]{} 0,\ \forall x\in\mathbb{R}$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

<u>Theorem</u> If $\exists\, M > 0$ such that $|f^{(n+1)}(t)| \leq M$ for all t, $a \leq t \leq x$, $\left(\overset{\text{or}}{x \leq t \leq a}\right)$ and <u>all $n \in \mathbb{N}$</u>

Then $$|R_n(x)| \leq \frac{M\,|x-a|^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0$$ and $\mathrm{Taylor}(f,a)(x) = f(x)$

Example * $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $x \in \mathbb{R}$

* For any $x \in \mathbb{R}$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

<u>Applications</u>. $\int_0^{\frac{1}{2}} \sin(t^2)\,dt = ?$

$$= \int_0^{\frac{1}{2}} \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \cdots\right) dt$$

$$= \frac{t^3}{3} - \frac{t^7}{7 \cdot 3!} + \frac{t^{11}}{11 \cdot 5!} - \cdots \Big|_0^{\frac{1}{2}}$$

$$\therefore \int_0^{\frac{1}{2}} \sin t^2 \, dt$$

$$= \frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{\left(\frac{1}{2}\right)^7}{7 \cdot 3!} + \frac{\left(\frac{1}{2}\right)^{11}}{11 \cdot 5!} - \cdots$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\left(\frac{1}{2}\right)^{4k-1}}{(4k-1)(2k-1)!}$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}\left(\frac{1}{2}\right)^{4k-1}}{(4k-1)(2k-1)!} + E_n$$

$$|E_n| \underset{k=n+1}{\le} \frac{\left(\frac{1}{2}\right)^{4n+3}}{(4n+3)(2n+1)!}$$

For example $|E_3| \le \frac{2^{-15}}{15 \cdot 7!} \simeq 4.04 \times 10^{-10}$