Taylor Series

Corollary (from term by term differentiation theorem):

If $f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$ converges on $|x-a|<R$, $\underline{\underline{R>0}}$, then $a_n = \frac{f^{(n)}(a)}{n!}$

$\underline{\text{proof}}$: $f^{(n)}(a) = \frac{d^n}{dx^n}\left(\sum \cdots\right)\Big|_{x=a} = n!\,a_n$

<u>Remark</u>: It is possible that $R=0$ (Example $\sum_{n=0}^{\infty} n!x^n$). In this case, $f(x)$ is only defined on $x=a$, ($f(a)=a_0$), but $f'(a), f''(a), \ldots$ are not defined.

<u>Def</u> Taylor Series generated by f at $x=a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\left[\stackrel{\text{def}}{=} \text{Taylor}(f, a)(x) \right]$$

Special case. $a=0$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \rightarrow \text{Mclaurin Series}$$

__Example__ $P(x) = $ polynomial

$$= \sum_{k=0}^{m} b_k x^k$$

$$\text{Taylor}(P, 0)(x) = P(x)$$

$$\frac{d^n}{dx^n}(x^k)\Big|_{x=0} = \begin{cases} k! & n=k \\ 0 & n \neq k \end{cases}$$

$$\frac{d^n}{dx^n}\left(\sum_{k=0}^{m} b_k x^k\right)\Bigg|_{x=0}$$

$$= \begin{cases} 0 & n > m \\ k!\,b_k = n!\,b_n & n = k \leq m \end{cases}$$

$\therefore \text{Taylor}(P, 0)(x)$

$$= \sum_{n=0}^{\infty} \frac{P^{(n)}(0)}{n!} x^n$$

$$= \sum_{n=0}^{m} \frac{n!\, b_n}{n!} x^n$$

$$= P(x)$$

Similarly,

$$\underline{\text{Taylor}(P, a)(x) = P(x)}$$

Example $f(x) = e^x$

$\mathrm{Taylor}(e^x, a)(x) = ?$

Ans: $\frac{d^n}{dx^n} e^x \Big|_{x=a} = e^a$

$$\therefore = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$$

$$= e^a \left(\sum_{n=0}^{\infty} \frac{1}{n!}(x-a)^n \right)$$

converges for any $x \in \mathbb{R}$

$(= e^x$? Yes, next time$)$

Example Taylor$(\cos x, 0)(x) = ?$

Ans: $\cos^{(n)}(0) =$

$$\begin{array}{cccc} 1, & 0, & -1, & 0, \dots\dots \\ \| & \| & \| & \| \\ \cos 0, & -\sin 0, & -\cos 0, & \sin 0 \\ n=0, & n=1, & n=2, & n=3 \end{array}$$

$$\text{Answer} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\dots$$

converges for any $x \in \mathbb{R}$

$(= \cos x$? Yes, next time$)$

Example $f(x)=\begin{cases}0 & x=0\\ e^{\frac{-1}{x^2}} & x\neq 0\end{cases}$

Note: $0\leq f(x)<1$

$$f(x)=0 \iff x=0$$

Does $f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}$ exist?

$$\left(\frac{0}{0}\right)\lim_{x\to 0}\frac{e^{\frac{-1}{x^2}}}{x}=\lim_{x\to 0}\frac{\frac{1}{x}}{e^{\frac{1}{x^2}}}\left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to 0}\frac{-x^{-2}}{-2x^{-3}e^{\frac{1}{x^2}}}=\lim_{x\to 0}\frac{x}{2e^{\frac{1}{x^2}}}\left(\frac{0}{\infty}\right)=0$$

$$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x}$$

$$= \lim_{x \to 0} \frac{2x^{-3} e^{\frac{-1}{x^2}} - 0}{x}$$

$$= \lim_{x \to 0} \frac{2x^{-4}}{e^{\frac{1}{x^2}}} \left(\frac{\infty}{\infty}\right)$$

$\underline{\underline{\text{l'Hôpital}}}$ (homework) $= 0$

In fact, $f^{(n)}(0) = 0$, $n = 0, 1, 2, \cdots$

Taylor(f,0)(x) $= 0$ function

$\neq f(x)$. if $x \neq 0$

This is an example where Taylor(f,0)(x) is NOT f(x), except when x=0.